Lecture 23: notes

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Defn. A power series is a series of the form,

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \dots + c_n x^n + \dots$$

where x is a variable and c_n's are constants.

A power series is a SERIES and a FUNCTION of x.

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

But the function above only makes sense at a given point x_0 (think $x_0 = 1$, $x_0 = -5$, etc) if the series,

$$f(x_0) = \sum_{n=0}^{\infty} c_n (x_0)^n$$

this is just a series NOT a function since I've plugged in a value for x.

converges.

Ex1. Does the power series, $\sum_{n=1}^{\infty} x^n$, converge at the point $x = \frac{1}{2}$?

Let's plug in the point $x=\frac{1}{2}$ to the power series: $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$. This is now a geometric series with $r=\frac{1}{2}$ $\Rightarrow$ the power series converges at $x=\frac{1}{2}$.

Ex2. Does the power series, $\sum_{n=0}^{\infty} x^n$, converge at $x=2$?

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (2)^n$$

The above diverges because the series is geometric with $r=2$ $\Rightarrow$ the power series diverges at $x=2$.

Suppose we had a power series $f(x)=\sum_{n=0}^{\infty} c_n x^n$ and we wanted to know ALL the values of x that the power series converged for.
To answer this question we compute the radius of convergence.

"all values of x that a power series converges for.

Ex3. Compute the radius of convergence of the power series $\sum_{n=0}^{\infty} x^n$.

Let's use the ratio test,

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{x}\right| = \lim_{n\to\infty}|x| = |x|$$

this doesn't depend on n.

The conclusion on the ratio test tells us that the power series converges for $|x|<1$, BUT the ratio test is inconclusive when $x=1$ & $x=-1$.

Let's test these points seperately,

$x=1$: $\sum (1)^n = \sum 1$ diverges

$x=-1$: $\sum (-1)^n$ diverges

So the radius of convergence is $R=1$ and the interval of convergence is $-1<x<1$.

Ex4. Compute the radius of convergence for the power series $\sum_{n=0}^{\infty} \frac{1}{3^n} x^n$.

Use the ratio test,

$$\lim_{n\to\infty} \left| \frac{x^{n+1}\cdot 3^n}{3^{n+1}\cdot x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3} \right| = \left| \frac{x}{3} \right| < 1 \Rightarrow |x|<3.$$

$x=3$: $\sum_{n=1}^{\infty} \frac{1}{3^n}(3)^n = \sum_1 1$ diverges.

$x=-3$: $\sum_1 \frac{1}{3^n}(-3)^n = \sum_1 (-1)^n$ diverges

So the radius of convergence is $R=3$ and the interval of convergence is $-3<x<3$.

In the above we've only been dealing with power series centered at $x=0$, so let's generalize this idea,

Defn. a series of the form,

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is a power series centered at a.

Ex5. For what values of x does $\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n$ converge?

Use the ratio test,

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1} \cdot n}{(n+1)\cdot(x-1)^n} \right| = \lim_{n\to\infty} |x-1| \cdot \frac{n}{n+1}$$

$$= |x-1| \Rightarrow |x-1| < 1$$

$$\Rightarrow -1 < x-1 < 1$$

Check the endpoints:

$(x-1) = 1$: $\sum_1 \frac{1}{n}(1)^n = \sum_1 \frac{1}{n}$ diverges

$(x-1) = -1$: $\sum_1 \frac{1}{n}(-1)^n = \sum_1 \frac{(-1)^n}{n}$ converges.

So $R=1$ and the power series converges for

$-1 \le x-1 < 1 \Rightarrow 0 \le x < 2$.

Ex 7. For what values of x does the power series $\sum_{n=0}^{\infty} \frac{1}{n!}(x+2)^n$ converge?

Use the ratio test,

$$\lim_{n\to\infty} \left| \frac{(x+2)^{n+1} \cdot n!}{(n+1)! \cdot (x+2)^n} \right| = |x+2| \lim_{n\to\infty} \frac{1}{n+1} = |x+2| \cdot 0$$

$$= 0$$

??

This will be 0 (and thus convergent) for <u>any</u> value of x.

$\Rightarrow$ $R=\infty$ and the power series converges for all x.