WS 12 solutions

Wednesday, July 24, 2019 10:39 AM

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+\sqrt{n}}$

Check absolute convergence:

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+\sqrt{n}}} = \lim_{n\to\infty} \frac{n+\sqrt{n}}{n} = 1.$$

Since $\sum_1 \frac{1}{n}$ diverges $\Rightarrow \sum_1 \frac{1}{n+\sqrt{n}}$ diverges by the limit comparison test $\Rightarrow \sum_1 (-1)^{n+1} \frac{1}{n+\sqrt{n}}$ does not absolutely converge.

Check conditional convergence:

1. $\lim_{n\to\infty} \frac{1}{n+\sqrt{n}} = 0$

2. $n+\sqrt{n} < n+1+\sqrt{n+1} \Rightarrow \frac{1}{n+\sqrt{n}} > \frac{1}{n+1+\sqrt{n+1}}$

So $\sum_1 (-1)^{n+1} \frac{1}{n+\sqrt{n}}$ $\boxed{\text{conditionally converges}}$ by the AST.

2. $\sum_{n=1}^{\infty} \frac{2^{2n}\cdot n!}{5^{n+1}(n+1)!} = \sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}(n+1)}$

Check absolute convergence = converges:

$$\lim_{n\to\infty} \frac{2^{2n+3}\cdot 5^{n+1}(n+1)}{5^{n+2}(n+2)\cdot 2^{2n+1}} = \lim_{n\to\infty} \frac{4(n+1)}{5(n+2)} = \frac{4}{5}$$

So the series $\boxed{\text{absolutely converges}}$ by the ratio test.

3. $\sum_{n=2}^{\infty} \frac{1}{\log(n!)}$

Check for absolute convergence:

$$\frac{1}{\log(n!)} = \frac{1}{\log 1 + \log 2 + \dots + \log n} \geq \frac{1}{(n+1)\log n}$$

Now we must check that $\sum_{n=2}^{\infty} \frac{1}{(n+1)\log n}$ diverges. Lets compare,

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)\log n}}{\frac{1}{n\log n}} = \lim_{n\to\infty} \frac{n \cancel{\log n}}{(n+1)\cancel{\log n}} = 1$$

I don't show this step but we've done it.

Since $\sum \frac{1}{n\log n}$ diverges by integral

Since $\Sigma_1 n\log n$ diverges by integral

test $\Rightarrow \Sigma_1 \frac{1}{(n+1)\log n}$ diverges by LCT $\Rightarrow$

$\Sigma_1 \frac{1}{\log(n!)}$ $\boxed{\text{diverges}}$ by comparison test.

4. $\Sigma_{n=1}^{\infty} \cos(n\pi)\cos\left(\frac{\pi}{2}+\frac{1}{n}\right) = \Sigma_1 (-1)^n \sin\left(\frac{1}{n}\right)$

Check for absolute convergent:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

Since $\Sigma_1 \frac{1}{n}$ diverges $\Rightarrow \Sigma \sin\left(\frac{1}{n}\right)$ diverges

by the LCT.

Check for conditional convergence:

1. $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$

2. decreasing (draw the graph).

So $\Sigma_1 (-1)^n \sin\left(\frac{1}{n}\right)$ $\boxed{\text{conditionally converges}}$

by the AST.

5. $\sum_{n=1}^{\infty} \frac{5^n+2^n}{4^n+3^n}$

Check for absolute convergence = convergence:

$$\lim_{n\to\infty} \frac{\frac{5^n+2^n}{4^n+3^n}}{\frac{5^n}{4^n}} = \lim_{n\to\infty} \frac{1+\left(\frac{2}{5}\right)^n}{1+\left(\frac{3}{4}\right)^n} = 1$$

$\sum_1 \left(\frac{5}{4}\right)^n$ diverges $\Rightarrow$ $\sum_1 \frac{5^n+2^n}{4^n+3^n}$ $\boxed{\text{diverges}}$

by LCT.

6. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

Check absolute convergence = convergence:

$$\lim_{n\to\infty} \frac{3^{n+1}(n+1)^2 \cdot n!}{(n+1)!\,3^n n^2} = \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)n^2} = 0$$

So by the ratio test it $\boxed{\text{absolutely converges.}}$

7. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$

Check absolute convergence = convergence:

$$\lim_{n\to\infty} \frac{\frac{e^{1/n}}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} e^{1/n} = 1$$

So the series $\boxed{\text{converges absolutely}}$ by the LCT.

8. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$

Check for absolute convergence = convergence:

$$\lim_{n\to\infty} \sqrt[n]{2}-1 = 0$$

So the series $\boxed{\text{converges absolutely}}$ by the root test.

9. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n)!}{n^n}$

Check for absolute convergence:

$$\lim \frac{(2n+2)!\cdot n^n}{} = \lim \frac{(2n+2)(2n+1)}{} \cdot \left(\frac{n}{}\right)^n$$

$$\lim_{n\to\infty} (n+1)^n \cdot (2n)! \qquad \lim_{n\to\infty} \frac{\quad}{(n+1)} \qquad (n+1)$$

$$= \infty$$

So by the ratio test the series $\boxed{\text{diverges.}}$

10. $\sum_{n=1}^{\infty} \frac{\sqrt{n}(1+n)}{n^2+3^n}$

Check for absolute convergence:

$$\frac{\sqrt{n}(1+n)}{n^2+3^n} \leq \frac{\sqrt{n}(1+n)}{3^n}$$

← use the ratio test on this.

$$\lim_{n\to\infty} \frac{\sqrt{n+1}(2+n)\cdot 3^n}{3^{n+1}\cdot\sqrt{n}(1+n)} = \lim_{n\to\infty} \frac{1}{3}\cdot\frac{\sqrt{n+1}(2+n)}{\sqrt{n}(1+n)} = \frac{1}{3}.$$

By the ratio test, $\sum_1 \frac{\sqrt{n}(1+n)}{3^n}$ converges

$\Rightarrow$ by the direct comparison test $\sum_1 \frac{\sqrt{n}(1+n)}{n^2+3^n}$

$\boxed{\text{converges absolutely.}}$