Lecture 21: notes

Monday, July 22, 2019 9:27 PM

<u>Def'n</u>. A series $\sum_i a_n$ is ==absolutely convergent== if the series $\sum_i |a_n|$ converges.

take the absolute value of the terms

<u>Ex 1</u>. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge absolutely?

The series absolutely converges if the series

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by the p-test),

the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ absolutely converges.

<u>Ex 2</u>. Does series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ absolutely converge?

The series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the p-test) so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does NOT

absolutely converge

absolutely converge.

Note that if a series $\sum a_n$ has positive terms, then absolute convergence and convergence are the same thing.

Def'n. a series $\sum a_n$ is conditionally convergent if it converges, but does not absolutely converge.

Only alternating series can conditionally converge!

Ex3. Does the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$ absolutely converge or conditionally converge?

Let's look at whether it absolutely converges

first,

☑ continuous

☑ positive

☑ decreasing

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \log n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \log n} \rightsquigarrow \int_2^{\infty} \frac{1}{x \log x} dx$$

$u = \log x$

$du = \frac{1}{x} dx$

$$= \int_{\log 2}^{\infty} \frac{1}{u} du$$

So we see that $\sum \left| \frac{(-1)^n}{n \log n} \right|$ diverges (by the integral test).

However $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$ converges (by the AST)

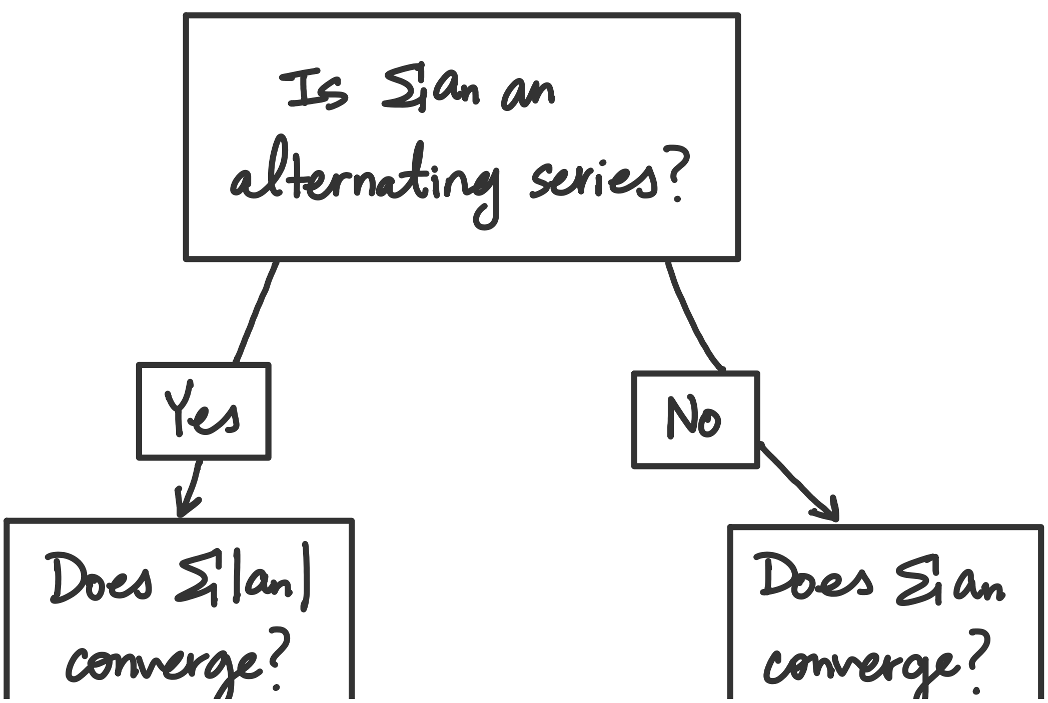
1. $\lim\limits_{n \to \infty} \frac{1}{n \log n} = 0$

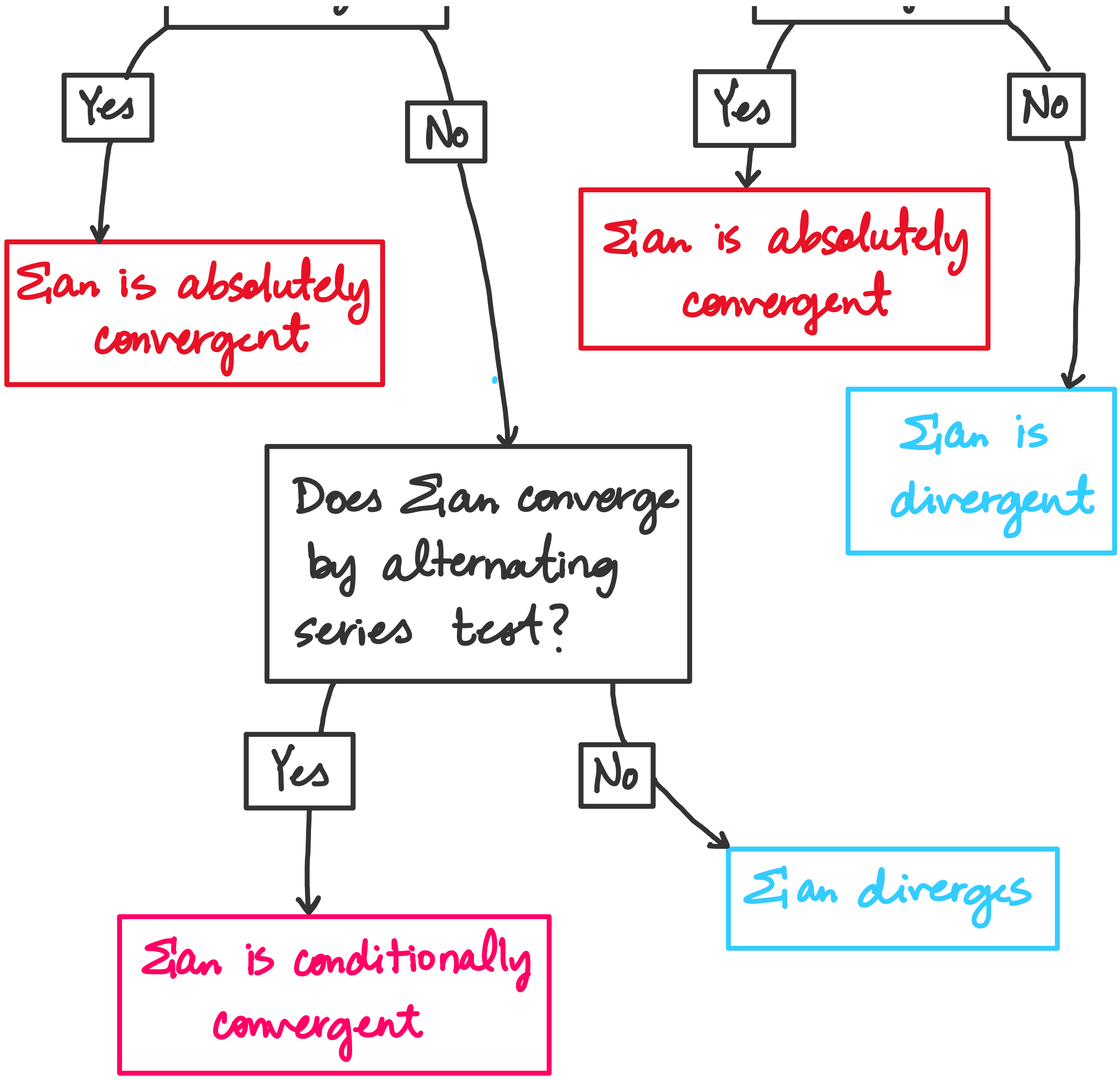
2. $n \log n < (n+1) \log (n+1)$

$\Rightarrow \frac{1}{n \log n} > \frac{1}{(n+1) \log (n+1)}$

So $\sum \frac{(-1)^n}{n \log n}$ conditionally converges.

Given a series $\sum a_n$ here are the steps to determine absolute convergence or conditional convergence.





Let's introduce some new test,

<u>The ratio test.</u>

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$,

then the series $\sum a_n$ diverges.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the ratio

test is inconclusive.

<u>Ex4</u>. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$a_n = \frac{n^2}{2^n}, \quad a_{n+1} = \frac{(n+1)^2}{2^{n+1}} = \frac{n^2+2n+1}{2\cdot 2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{n^2+2n+1}{2\cdot 2^n}}{\frac{n^2}{2^n}}$$

what's in numerator matters here!
Don't flip the fraction!

$$= \lim_{n\to\infty} \frac{\cancel{2^n}(n^2+2n+1)}{2\cdot \cancel{2^n}\cdot n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2}$$

Since $L = \frac{1}{2}$, the series is absolutely convergent by the ratio test.

The ratio test has its limits though!

The ratio test has its limits though:

Ex5. $\sum \frac{1}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1. \quad ☹$$

The ratio test is inconclusive!

The root test.

(i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ absolutely converges.

(ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

Ex 6. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n \Longrightarrow \sqrt[n]{|a_n|} = \frac{2n+3}{3n+2}$$

$$\lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3}$$

Since $L < 1$ the series is absolutely convergent.