Lecture 20: notes

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<u>Defn</u>. An alternating series is a series of the form,

$$\sum_{1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{1}^{\infty}(-1)^{n+1} b_n$$

where $b_n > 0$ for all n. In other words, the series alternates between positive and negative values.

<u>Note</u>: most of the tests of convergence we have dealt with require the terms in the series to be positive, this is because the criteria for convergence is different when a series alternates.

<u>Ex 1</u>. Does the series $\sum_{1}^{\infty} \frac{(-1)^n}{n}$ converge?

the alternating harmonic series

Let's write out some partial sums:

$s_1 = -1$

(*) $s_2 = -1 + \frac{1}{2} = -\frac{1}{2}$

$s_3 = -1 + \frac{1}{2} - \frac{1}{3} = \frac{-5}{6}$

(*) $s_4 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = -\frac{7}{12} \sim -0.58$

$s_5 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} = \frac{-47}{60} \sim -0.78$

(*) $s_6 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = -\frac{37}{60} \sim -0.62$

$s_7 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} = \frac{-319}{420} \sim -0.76$

the number seem to be converging to some number around -0.7 (in fact it's $-\log(2)$).

Let's prove this converges by considering the sequence of even partial sums $\{S_{2n}\}_{n\geq 1}$.

Let's show this sequence is bounded and monotonic. ← decreasing

Bounded:

$$S_{2n} = \underbrace{(-1+1/2)}_{\ominus} + \underbrace{(-1/3+1/4)}_{\ominus} + \ldots + \underbrace{(-1/(2n-1) + 1/2n)}_{\ominus}$$

So the sequence $\{S_{2n}\}$ is bounded above by 0.

$$S_{2n} = -1 + \underbrace{(1/2 - 1/3)}_{\oplus} + \underbrace{(1/4 - 1/5)}_{\oplus} + \ldots + \underbrace{(1/(2n-2) - 1/(2n-1))}_{\oplus} + \underbrace{1/2n}_{\oplus}$$

The above tells us the sequence is bounded below by -1.

So $-1 \leq S_{2n} \leq 0$ for all n.

Monotonic:

$$S_{2n} = (-1+\tfrac{1}{2}) + (-\tfrac{1}{3}+\tfrac{1}{4}) + \ldots + \left(\tfrac{1}{2n-1} - \tfrac{1}{2n}\right)$$

$$S_{2n} = \qquad S_{2n-1} \qquad + (1/(2n-1) - 1/2n)$$

From the above we see $S_{2n} = S_{2n-1} +$ positive number

$S_{2n} \geq S_{2n-1}$.

An so the sequence is monotonic decreasing.

So $\{S_{2n}\}$ is monotonic & bounded $\Rightarrow$ convergent thus,

$$\lim_{n\to\infty} S_{2n} = S$$

We can't yet conclude that $\lim_{n\to\infty} S_n = S$ as well, we need to check the subsequence of odd terms, $\{S_{2n+1}\}$.

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} - \frac{1}{2n+1} = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty}\frac{1}{2n+1}$$

$$= s - 0$$

$$= s$$

Since $\{s_{2n}\}\to s$ and $\{s_{2n+1}\}\to s$ we can conclude $\lim_{n\to\infty} s_n = s \Rightarrow \sum_1 (-1)^n \frac{1}{n}$ converges. $\square$

That was a lot of work, let's state a test to make it easier.

<u>Alternating series test</u>. Suppose we have a series $\sum_1 a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

(1) $\lim_{n\to\infty} b_n = 0$

(2) $\{b_n\}$ is a decreasing sequence

the series $\sum_1 a_n$ converges.

<u>Ex2</u>. Does the series $\sum_1 \frac{(-1)^{n+1}}{\sqrt{n}}$ converge or diverge?

In this example $b_n = \frac{1}{\sqrt{n}}$.

1. Show decreasing: $\underbrace{\sqrt{n} < \sqrt{n+1}}_{\text{(obvious)}} \Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$

2. Show $\{b_n\}\to 0$: $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Ex3. D... $\sum^{\infty} \frac{(-1)^n}{\cdots}$?

Ex3. Does the series $\sum_{n=1} \frac{1}{7+2n}$ converge?

1. Check decreasing: $7+2n < 7+2(n+1) \Rightarrow \frac{1}{7+2n} > \frac{1}{7+2(n+1)}$

2. Check limit: $\lim_{n\to\infty} \frac{1}{7+2n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{7}{n}+2}$

$$= \frac{0}{0+2} = 0.$$

So $\sum_1 (-1)^n \cdot \frac{1}{7+2n}$ converges.

The alternating series test only tells us if an alternating series converges. To prove divergence, use the divergence test.

Ex4. Does the series $\sum_1 (-1)^n \cdot \frac{n^2}{n^2+3}$ converge or diverge?

Let's look at the limit,

$$\lim_{n\to\infty} \frac{n^2}{n^2+3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n^2}}$$

$$= 1 \Rightarrow \lim_{n\to\infty} (-1)^n \frac{n^2}{n^2+3} \neq 0$$

$\Rightarrow$ diverges.

Ex5. Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$ converge or diverge?

condition for decreasing

1. determine decreasing:

$$\left(\frac{\log n}{n}\right)' = \frac{n \cdot \frac{1}{n} - \log n}{n^2} = \frac{1-\log n}{n^2} < 0$$

$$\Rightarrow 1 - \log n < 0$$

$$\Rightarrow \log n > 1 \Rightarrow n > e.$$

so the sequence $\{b_n = \log n / n\}$ is decreasing for all $n \geq 3$.

2. Check limit: $\lim\limits_{n\to\infty} \frac{\log n}{n} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{1}$

$$= \lim_{n\to\infty} \frac{1}{n} = 0.$$