# Lecture 19: notes

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Sometimes the comparison test causes silly issues. Like the following,

Ex1. Does the series $\sum_{n=1}^{\infty} \frac{1}{|n^2-30|}$ converge or diverge?

The "dominating term" is $\frac{1}{n^2}$, so our guess is that the above converges, however

$$\frac{1}{|n^2-30|} \cancel{\leq} \frac{1}{n^2}$$

we could multiply by a constant, but the limit comparison test will help us get around this.

Limit Comparison test. Let $\sum a_n$ and $\sum b_n$ be non-negative series such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \longleftarrow \begin{matrix} c>0 \\ c\neq\infty \end{matrix}$$

Then either both series converge or both series diverge.

Proof.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \Rightarrow m \leq \frac{a_n}{b_n} \leq M$$

$$\Rightarrow m b_n \leq a_n \leq M b_n$$

$$\Rightarrow \underbrace{\sum_1 m b_n \leq \sum_1 a_n}_{} \underbrace{\leq \sum_1 M b_n}_{}$$

if $\sum_1 b_n$ div,
$\sum_1 m b_n = \infty \Rightarrow \sum_1 a_n = \infty$

$\sum_1 b_n$ conv,
$\sum_1 M b_n = s \geq \sum_1 a_n = s'$.

Note that the choice of $a_n$ & $b_n$ was arbitrary, so if we know whether $\sum_1 a_n$ conv or div we can make conclusions about $\sum_1 b_n$.

Ex 1. cont'd

So let's limit compare the series $\sum_1 \frac{1}{n^2}$ and $\sum_1 \frac{1}{|n^2-30|}$,

$$\lim_{n\to\infty} \frac{\frac{1}{|n^2-30|}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{|n^2-30|}$$

for large n abs. value does nothing

$$= \lim_{n\to\infty} \frac{n^2}{n^2-30} = 1.$$

Why is it so important that $c \neq 0$ and $c \neq \infty$? Let's see why,

$\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$ $\rightsquigarrow$ $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} n = \infty$

diverges. converges.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\sum \frac{1}{n^2}$ & $\sum \frac{1}{n^3}$ $\rightsquigarrow$ $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} n = \infty$

converge converge

$$\lim_{n\to\infty} \frac{\frac{1}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

We can see that anything is possible when the limit is 0 or $\infty$.

Ex2. Does the series $\sum_{n=1}^{\infty} \frac{1}{3^n - n + 1}$ converge or diverge?

Let's compare to the convergent series $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

$$\lim_{n\to\infty} \frac{\frac{1}{3^n}}{\frac{1}{3^n - n + 1}} = \lim_{n\to\infty} \frac{3^n - n + 1}{3^n}$$

$$= \lim_{n\to\infty} \frac{\log 3 \cdot 3^n - 1}{\log 3 \cdot 3^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(\log 3)^2} \cdot \cancel{3^n}}{\cancel{(\log 3)^2} \cdot \cancel{3^n}} = 1.$$

So by the limit comparison test $\sum_1 \frac{1}{3^n - n + 1}$ also converges.

Ex3. Does the series $\sum_{n=1}^{\infty} \frac{3^n}{2^n + n}$ converge or diverge?

Let's use the limit comparison test with the divergent series $\sum_1 \left(\frac{3}{2}\right)^n$.

$$\lim_{n\to\infty} \frac{\left(\frac{3}{2}\right)^n}{\frac{3^n}{2^n+n}} = \lim_{n\to\infty} \frac{\cancel{3^n}(2^n+n)}{\cancel{3^n}\cdot 2^n}$$

$$= \lim_{n\to\infty} \frac{2^n + n}{2^n} = 1$$

we could prove this just like ex 2.

Careful! Sometimes we can't find anything useful to compare to. If you get a limit of $c=0$ or $c=\infty$, try to use a different test.

$\sum^{\infty} \frac{\sqrt{n^2+2}}{\ldots}$

Ex 4. $\sum_{n=1} \frac{\quad}{n^4-n+3}$

Compare to $\sum \frac{n}{n^4} = \sum \frac{1}{n^3}$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^2+2}}{n^4-n+3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3\sqrt{n^2+2}}{n^4-n+3}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^8+2n^6}}{n^4-n+3} \times \frac{\frac{1}{n^4}}{\frac{1}{n^4}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+2/n^2}}{1-1/n^3+3/n^4} = 1.$$

all these $\to 0$ as $n \to \infty$.