Lecture 16: notes

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Topics on the exam:

1. Improper integrals
   i. infinite integrals
   ii. singularities
2. Applications of integration
   i. length of curve
   ii. surface area of solid of revolution
3. Convergence of sequence
4. Series
   i. geometric series
   ii. telescoping series
   } these tell us the sum of the series
   iii. integral test. — this doesn't tells us the sum.

Ex1. (True or false) if the sequence $\{a_n\}$ converges then the sequence $\{b_n = (-1)^n a_n\}$ also converges.

Let's assume $a_n \geq 0 \ \forall n$.

Since $\{b_n\}$ is an alternating sequence then it converges if

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} |b_n| = 0$$

The above implies,

$$\lim_{n\to\infty} a_n = 0$$

And so the statement is FALSE. A counter-ex would be $a_n = 1$ $\forall n$ (thus $\{b_n = (-1)^n\}$).

this doesn't converge.

Ex2. (True or false) The integral $\int_1^\infty x^p\,dx$ converges for all $p<-1$.

This is TRUE. The above is just the p-test stated differently.

$\frac{1}{x^p},\ p>1 \Rightarrow x^p,\ p<-1.$

Ex3. Does the integral $\int_{10}^\infty \frac{\ln x}{x^3+x}\,dx$ converge or diverge?

I'm going to guess the integral converges because of the x^3 in the denominator. This looks too hard to integrate, so let's use the comparison test.

$\log x \le x$ for $x \geq 10$

$$\frac{\log x}{x^3+x} \le \frac{\log x}{x^3} \le \frac{x}{x^3} = \frac{1}{x^2}$$

$\int_{10}^\infty \frac{1}{x^2}\,dx$ converges by the p-test $\Rightarrow$ $\int_{10}^\infty \frac{\log x}{x^3+x}\,dx$ converges by the comparison test

Ex4. Does the sequence $\left\{a_n = (-1)^n \frac{\sin n}{n}\right\}$ converge?

The alternating sequence converges iff $\lim_{n\to\infty} |a_n| = 0$.

Note that $|a_n| = |\sin n|/n$

$0 \le |a_n| = \frac{|\sin n|}{n} \le \frac{1}{n}$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} |a_n| = 0$ by squeeze thm

Ex5. Compute the surface area of the solid created by rotating $y=\sqrt{x}$, $2 \le x \le 4$ about the x-axis.

Use the formula $S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$,

$f(x) = \sqrt{x}$, $\quad a=2$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad b=4$

$$S = 2\pi \int_2^4 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx$$

$$= 2\pi \int_2^4 \sqrt{x+\frac{1}{4}}\,dx \qquad u = x+\frac{1}{4}, \; du = dx$$

$$= 2\pi \int_{9/4}^{17/4} \sqrt{u}\,du = 2\pi \left[\frac{2}{3}u^{3/2}\right]_{9/4}^{17/4}$$

$$= \frac{4}{3}\pi \left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right]$$

Ex6. Does the series $\sum_{n=1}^{\infty} \frac{1}{1+e^n}$ converge or diverge?

Let's use the integral test, $f(x) = \frac{1}{1+e^x}$.

☑ continuous ? ... = means "want to show"

- continuous
- ☑ positive (obvious)
- ☑ decreasing: WTS $f'(x) \overset{?}{<} 0$

$$\frac{-e^x}{(1+e^x)^2} \overset{?}{<} 0$$

$$-e^x < 0$$ ✓ this is true for all x

$$\int_1^\infty \frac{1}{1+e^x}\,dx \leq \int_1^\infty \frac{1}{e^x}\,dx$$

$$= \int_1^\infty e^{-x}\,dx \leftarrow \text{converges}$$

$$\Rightarrow \int_1^\infty \frac{1}{1+e^x}\,dx \text{ converges} \Rightarrow \sum_{n=1}^\infty \frac{1}{1+e^n} \text{ converges.}$$

Ex 7. Determine whether the following series, $\sum_{n=1}^\infty \log\left[\frac{n}{n+2}\right]$, converges or diverges

Let's use telescoping to find a formula for the sequence of partial fractions, s_n:

$$\sum_i \log\left(\frac{n}{n+2}\right) = \sum_i \left(\log n - \log(n+2)\right)$$

$$s_1 = \cancel{\log 1}^{\,0} - \log 3$$

$$s_2 = (\cancel{\log 1}^{\,0} - \log 3) + (\log 2 - \log 4)$$

$$s_3 = (\cancel{\log 1}^{\,0} - \cancel{\log 3}) + (\log 2 - \log 4) + (\cancel{\log 3} - \log 5)$$

$$s_8 = (\cancel{\log 1} - \cancel{\log 3}) + (\log 2 - \cancel{\log 4}) + (\cancel{\log 3} - \cancel{\log 5})$$
$$+ (\cancel{\log 4} - \cancel{\log 6}) + (\cancel{\log 5} - \cancel{\log 7}) + (\cancel{\log 6} - \cancel{\log 8})$$

$$+(\cancel{\log 7} - \log 9) + (\cancel{\log 8} - \log 10)$$

$$s_8 = \log 2 - \log 9 - \log 10 \Rightarrow s_n = \log 2 - \log((n+1)(n+2))$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \log 2 - \log[(n+1)(n+2)] = -\infty$$

sequence of partial sums diverges.

$$\Rightarrow \sum \log\left(\frac{n}{n+2}\right) \quad \text{diverges}$$