Lecture 15: notes

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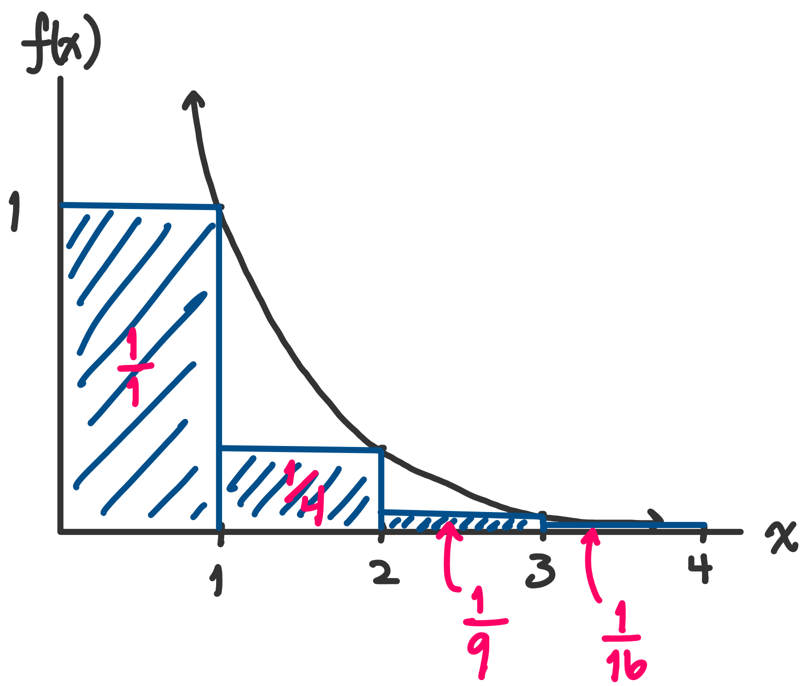
Consider the function $f(x) = \frac{1}{x^2}$, we know from chapter 7 that

$$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

So consider the following picture,



We can see from the picture that,

$$\int_1^\infty \frac{1}{x^2}\,dx \geq \sum_{n=2}^{\infty} \frac{1}{n^2} \geq 0$$

doesn't include the first rectangle.

So the above implies $\sum^{\infty} \frac{1}{n^2}$ converges

So the above implies $\sum_{n=1}^{\infty} n^{-}$ converges and so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges.

<u>The integral test</u>. Let $f(x)$ be a continuous, positive, decreasing function on $[1, \infty)$ and let $f(n) = a_n$. Then,

(i) $\int_1^{\infty} f(x)\,dx$ convergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ convergent

(ii) $\int_1^{\infty} f(x)\,dx$ divergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ divergent.

<u>Thm</u>. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$.

<u>Ex 1</u>. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges.

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2} \neq \int_1^{\infty} \frac{1}{x^2}\,dx$.

<u>Ex 2</u>. $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

positive

$f(x) = \frac{1}{x\log x}$ is decreasing & pos. for $x \geq e$, so we can apply the integral test,

$$\int_1^\infty \frac{1}{x\log x}\,dx = \int_{x=1}^\infty \frac{1}{u}\,du \quad \text{diverges.}$$

$\Rightarrow \sum_{n=2}^\infty \frac{1}{n\log n}$ diverges.

Suppose $\sum_{n=1}^\infty a_n$ converges, but we do not know its sum, $\sum_{n=1}^\infty a_n = s$. We can estimate s using,

$$\boxed{s_n + \int_{n+1}^\infty f(x)\,dx \leq s \leq s_n + \int_n^\infty f(x)\,dx}$$

the larger the n we pick, the better the estimate we will get.

Ex 3. Estimate the value of $\sum_{n=1}^\infty \frac{1}{n^2}$ using $n=2$.

$$s_2 = \frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4}$$

$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{x}\right]_2^t = \frac{1}{2}$$

$$\int_3 \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_3 \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{-1}{x}\right]_3^t = \frac{1}{3}$$

$$\frac{5}{4} + \frac{1}{3} \le s \le \frac{5}{4} + \frac{1}{2} \Rightarrow \frac{19}{12} \le s \le \frac{7}{4}.$$