Lecture 14: notes

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Consider the sequence $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$, we can easily write out the first few terms,

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

We can even sum up the first few terms,

$$S_1 = \frac{1}{2}$$
$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$
$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$
$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$$

The above begs the question: what happens if we sum up all of the terms of the sequence?

What does it mean to add up infinitely many things?
Does the sum even exist?

We call

We write,

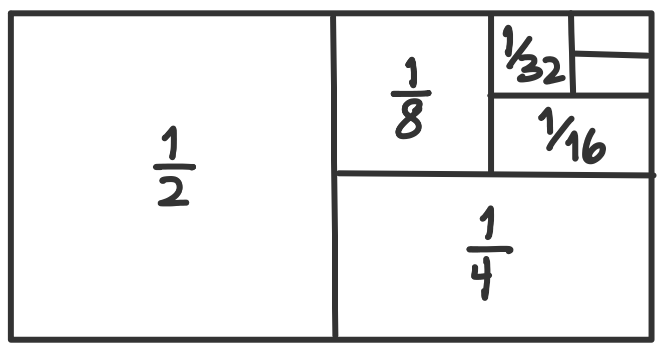
this a series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}+\dots = \sum_{n=1}^{\infty}\frac{1}{2^n}$$

this is just the limit of a sequence, but a sequence that's hard to write down

$$= \lim_{n\to\infty} s_n$$

Let's draw a picture to help us understand if this sum exist,



From the picture we can see that,

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{2^n} = 1}$$

Def'n. Given a series $\sum_{n=1}^{\infty} a_n = a_1+a_2+\dots+a_n+\dots$ we say the series is convergent if $\sum_{n=1}^{\infty} a_n = s$ for some finite number s.

Denote the nth partial sum by, $s_n = a_1+a_2+\dots+a_n$.
If the sequence of partial sums $\{s_n\}$ converges

If the sequence of partial sums $\{S_n\}$ converges AS A SEQUENCE, then the series $\sum_{n=1}^{\infty} a_n$ converges AS A SERIES.

Example 1. Geometric series, $\sum_{n=1}^{\infty} ar^{n-1}$.

Let's try to find a formula for the nth partial sum,

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$
$$= a(1 + r + r^2 + r^{n-1})$$
$$S_n = a\frac{(1-r^n)}{1-r}$$

So $\sum_{n=1}^{\infty} ar^{n-1}$ converges iff $\left\{S_n = a\frac{(1-r^n)}{1-r}\right\}_{n=1}^{\infty}$ converges

Compute when this converges as an exercise.

For $-1 < r < 1$ the series converges and,

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} a\frac{(1-r^n)}{1-r} = \frac{a}{1-r}}$$

Note that most of the time we will be unable to compute the exact sum. For geometric series we can though.

<u>Example</u> 2. Compute the sum, $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$.

$r = \frac{1}{2} \Rightarrow$ convergent geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} = 1$$

<u>Thm</u>. (Test for divergence) If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Example</u> 3. Prove that the series $\sum_{n=1}^{\infty} 3n+2/5n$ diverges

$$\lim_{n\to\infty} \frac{3n+2}{5n} = \lim_{n\to\infty} \frac{3n}{5n} = \frac{3}{5} \neq 0 \Rightarrow \sum_{n=1}^{\infty} 3n+2/5n$$ diverges

<u>Thm</u>. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series then so are $\sum_1 ca_n$, $\sum_1 (a_n+b_n)$, and $\sum_1 (a_n-b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c\left(\sum_{n=1}^{\infty} a_n\right)$

(ii) $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

Example 4. Find the sum of $\sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n \right]$

We see there are two convergent geometric series, so

$$\sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n \right] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}} = 2 + \frac{3}{2}$$