Lecture 13: notes

Monday, July 8, 2019 3:45 PM

Def'n. A sequence is bounded above if $\exists M$ such that $a_n \le M \;\forall n \geq 1$.
A sequence is bounded below if $\exists m$ such that $a_n \geq m \;\forall n \geq 1$.
If a sequence $\{a_n\}$ is bounded above and below then it is a bounded sequence.

Def'n. A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$. It is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.
A sequence that is increasing or decreasing is called monotonic.

Thm. (Monotonic sequence) Every bounded monotonic sequence is convergent.

Ex1. Prove $\{a_n\}$ converges, $a_1 = 1$, $a_{n+1} = \frac{1}{2}(a_n + 2)$.

Step 1. Show $\{a_n\}$ is monotonic.

We will use mathematical induction,

1. Check $a_1 < a_2$

$$1 < 3/2 \checkmark$$

2. Assume $a_n < a_{n+1}$

3. Prove $a_{n+1} < a_{n+2}$:

$$a_n < a_{n+1}$$
$$a_n + 2 < a_{n+1} + 2$$
$$\frac{1}{2}(a_n + 2) < \frac{1}{2}(a_{n+1} + 2)$$
$$a_{n+1} < a_{n+2}$$

This proves monotonicity.

$\boxed{\text{Step 2.}}$ Prove $\{a_n\}$ is bounded.

It is obvious that $a_n > 0 \ \forall n$.

Let's show that it's bounded above by 2, we will again use induction,

$$a_n < 2$$
$$a_n + 2 < 4$$
$$\frac{1}{2}(a_n + 2) < 2$$

$a_{n+1} < 2$.

This proves that $\{a_n\}$ is bounded.

So $\{a_n\}$ converges by the above theorem.

Now let's see some examples of interesting sequences.

Ex2. Fibonacci numbers; $f_{n+2} = f_{n+1} + f_n$, $f_1 = f_2 = 1$.

$$f_1 = f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5, \ldots$$

① $$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = L$$

② $$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \frac{f_n + f_{n-1}}{f_n} = \lim_{n\to\infty} \frac{f_n}{f_n} + \lim_{n\to\infty} \frac{f_{n-1}}{f_n}$$
$$= 1 - \frac{1}{L}$$

From ① and ② we get,

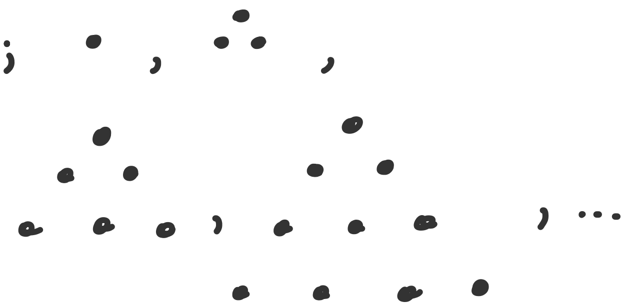
the golden ratio.

$$L = 1 + \frac{1}{L} \Rightarrow L^2 - L - 1 = 0$$
$$L = \frac{1 \pm \sqrt{5}}{2} \Rightarrow L = \frac{1+\sqrt{5}}{2} = \varphi$$

Fun fact: every positive integer can be written

Fun fact: every positive integer can be written as a sum of Fibonacci numbers, where any number is used at most once.

Ex3. Triangle numbers;



Let's try to write a closed formula,

$a_1 = 1$

$a_2 = 3 = 1+2$

$a_3 = 6 = 3+3 = a_2 + 3$

$a_4 = 10 = 6+4 = a_3 + 4$

$\vdots$

$$\boxed{a_n = a_{n-1} + n}$$