Lecture 12: notes

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Today we will learn all about sequences and the main question we will ask is: does a given sequence converge?

To address this question we will discuss the following:

1) What is a sequence?
2) How we express sequences & some common examples.
3) Convergence & divergence of sequences.
4) Properties of sequence limits.

What is a sequence

Def'n. A sequence is an ordered list of numbers, often infinite. Sequence terms are denoted as follows:

a_1 – first term of the sequence

a_2 – second term

$\vdots$

a_n – nth term

a_{n+1} – $(n+1)$th term

Expressing sequences

We can express sequences in a few different ways,

$a_1, a_2, a_3, a_4, \ldots$ list form

$\{a_1, a_2, a_3, \ldots\}$ list form w/ brackets

$\{a_n\}$ brackets

$\{a_n\}_{n=1}^{\infty}$ brackets w/ bounds

these all indicate the same thing

We can indicate terms in the sequence with an implicit expression or an explicit expression,

in terms of previous sequence terms

<u>Example 1</u>. $a_{n+1} = 3 + a_n$, $a_1 = -1$.

The above is an implicit equation, let's write out the first couple terms,

$a_1 = -1$

$a_2 = 2$

$a_3 = 5$

$a_4 = 8$

This is an example of an arithmetic sequence.

Example 2. $a_n = 3 \cdot (2)^{n-1}$

The above is an explicit equation. Let's write out the first few terms,

$a_1 = 3$

$a_2 = 6$

$a_3 = 12$

$a_4 = 24$

The above is an example of a geometric sequence.

Example 3. Write down the first few terms of the sequence $\{a_n\}$ given by, $a_n = (-1)^n \cdot \frac{1}{n+1}$.

$a_1 = -\frac{1}{2}$

$a_2 = \frac{1}{3}$

$a_3 = -\frac{1}{4}$

$a_4 = \frac{1}{5}$

Exercise. Write down the first 5 terms of the following:

a. $a_n = \frac{n^2}{n+1}$

b. $a_n = n - a_{n-1}$, $a_1 = 1$

c. $a_n = (-1)^{n+1}(n+1)^{-2}$

d. $a_{n+2} = a_{n+1} + 2a_n$, $a_1 = 1$, $a_2 = 1$

Convergence of sequences.

<u>Def'n.</u> a sequence **converges** if the limit as $n \to \infty$ is defined (& not infinite).

In other words if the limit of the sequence is some finite number.

<u>Def'n.</u> If the limit of a sequence is undefined or infinite the sequence **diverges.**

(*) BE CAREFUL! The precise meaning of convergence and divergence depends on the type of problem we are dealing with.

<u>Example 4.</u> Does the following sequence converge, $\left\{a_n = \dfrac{n+3}{n}\right\}_{n=1}^{\infty}$

Consider the function $f(x) = \dfrac{x+3}{x}$. It is clear that for $n \in \mathbb{N}$, $f(n) = a_n$.

So let's consider the limit as $x \to \infty$ of $f(x)$,

$$\lim_{x \to \infty} \frac{x+3}{x} \overset{\text{l'hopital's rule.}}{=} \lim_{x \to \infty} \frac{1}{1} = 1$$

The claim is that $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x) = 1$

this is true,
let's formalize it.

Properties of sequence limits

Sequence limits are very similar to function limits with a few nuances. Let's go over some properties,

Thm. Given a sequence $\{a_n\}$, if there exist a continuous function $f(x)$ such that $f(n) = a_n$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{n\to\infty} a_n$.

Example 5. Does the sequece $\{a_n = n^2\}$ converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \lim_{x\to\infty} x^2 = \infty$$

$\Rightarrow$ the sequence $\{a_n\}$ diverges.

Properties. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences, then

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

doesn't depend on n!

2. $\lim\limits_{n\to\infty} c a_n = c \cdot \lim\limits_{n\to\infty} a_n$, c is a constant

3. $\lim_{n\to\infty} a_n \cdot b_n = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right)$

4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, as long as $\lim_{n\to\infty} b_n \neq 0$.

5. $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$, as long as $a_n \geq 0$ and p does not depend on n.

Thm. (squeeze thm) If $a_n \leq c_n \leq b_n$ for all $n \geq N$ (any integer) and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$.

Thm. The sequence $\{r^n\}_{n=1}^{\infty}$ converges for all r such that $-1 < r \leq 1$. Also,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \end{cases}$$

Exercise. Determine whether the following sequences converge,

1. $a_n = \frac{e^n}{n}$

2. $a_n = (-1)^{n+1}$

4. $a_n = \frac{(-1)^n \cdot n}{n!}$

5. $a_n = \sqrt{n+1} - \sqrt{n}$

3. $a_n = \frac{3n-1}{2n^2+1}$

6. $a_n = \log\left(\frac{n+2}{3n-1}\right)$