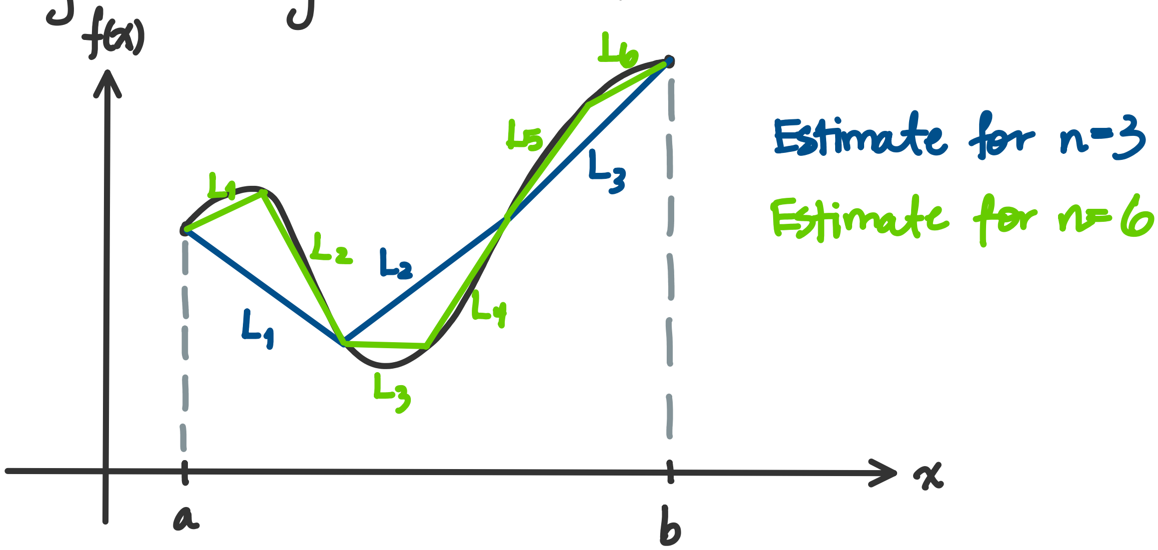
Lecture 11 notes

Saturday, July 6, 2019 9:05 PM

Today we will learn how to compute the length of a curve & the area of a surface of revolution.

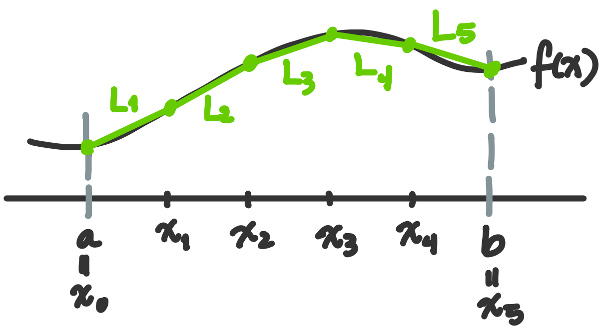
Arc length

We wish to compute the arc length of a curve to do so we will approximate the length using line segments and take the limit.



We see in the above that the estimate for n=6 is closer to the actual length than n=3, thus

$$\text{arc length} = \lim_{n\to\infty} \sum_{i=1}^{n} (\text{length } L_i)$$



$$\text{length } L_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

just the distance formula

$$= \sqrt{\left[1 + \left(\underbrace{\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}}_{\text{the slope of } L_i!}\right)^2\right](x_i - x_{i-1})^2}$$

$$= \sqrt{1 + f'(x_i^*)^2} \cdot \Delta x$$

by Mean Value Theorem this value is the slope of f(x) somewhere between x_{i-1} and x_i

So we can conclude that,

$$\text{arc length} = \lim_{n \to \infty} \sum_{i=1}^{n} (\text{length } L_i)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2} \cdot \Delta x$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Limit of Riemann sums

And we've proven the arc length formula.

$$\boxed{\text{arc length} = \int_a^b \sqrt{1+(f'(x))^2}\, dx}$$

Example 1. Determine the length of $y = \log(\sec x)$ between $0 \le x \le \frac{\pi}{4}$

Let's compute $f'(x)$ first,

$$f'(x) = \frac{1}{\sec x} \cdot \sec x \tan x$$
$$= \tan x$$

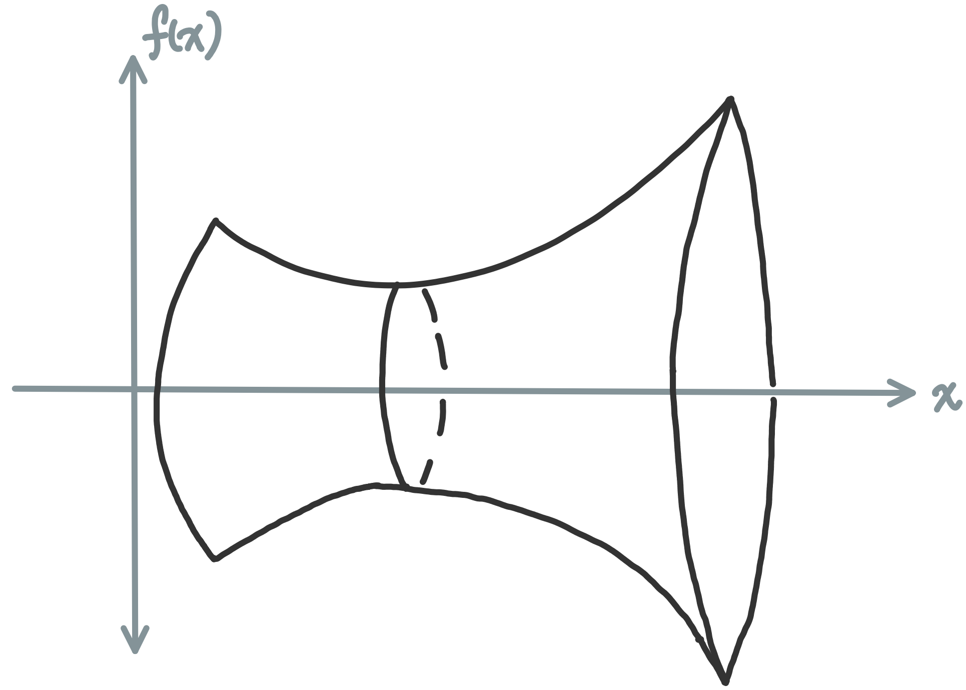
$$\text{arc length} = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx$$
$$= \int_0^{\pi/4} \sec x\, dx = \log|\sec x + \tan x|\Big]_0^{\pi/4}$$
$$= \log(\sqrt{2}+1) - \cancel{\log(1)}^{\,0}$$

Surface area of a solid of revolution

Previously we've seen how to compute the volume of a $\underbrace{\text{solid of revolution}}$, now let's compute its surface area.

created by rotating a

line around an axis



This is prove similarly to the last one, so we will skip the details and give the formula.

$$\boxed{\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx}$$

If we wanted to rotate around the y-axis we would use the formula,

$$S = \int_c^d 2\pi h(y)\sqrt{1+(h'(y))^2}\,dy$$

derivative WRT y
<u>not</u> x.

<u>Example 2</u>. Determine the surface area of the solid obtained by rotating $y=x^3$, $0 \le x \le 2$ around

the x-axis.

$$S = \int_0^2 2\pi x^3 \sqrt{1+(3x^2)^2}\, dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1+9x^4}\, dx \qquad u = 1+9x^4 \quad du = 36x^3\, dx$$

$$= \frac{2\pi}{36} \int_{x=0}^{2} \sqrt{u}\, du = \frac{2\pi}{36} \cdot \left[\frac{2}{3} u^{3/2} \right]_{x=0}^{2}$$

$$= \frac{\pi}{27} (1+9x^4)^{3/2} \Big]_{x=0}^{2}$$

$$= \frac{\pi}{27} \left[(1+144)^{3/2} - 1 \right] = \frac{\pi}{27} \left[145^{3/2} - 1 \right]$$