# Lecture 10: notes

Friday, July 5, 2019 9:30 AM

Today we are going to ask the following question,

> Without solving an improper integral, can we determine if it converges or diverges?

We will only address this question for infinite integrals and to start we will consider the following question,

<u>Example 1.</u> For what values of p is the following integral convergent,

$$\int_1^\infty \frac{1}{x^p}\,dx$$

We will solve the above using the procedure from Friday,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$$

$$= \lim_{t\to\infty} \left[ \frac{1}{1-p} \cdot \frac{1}{x^{p-1}} \right]_1^t$$

for $p \neq 1$!

compute seperately for $p = 1$.

$$= \lim_{t \to \infty} \frac{1}{1-p} \left( \frac{1}{t^{p-1}} - 1 \right)$$

plug in certain values of p (except 1)

For $p = 1$:

$$\int_1^\infty \frac{1}{x} dx = \lim_{t \to \infty} \int_1^t \frac{1}{x} dx$$

$$= \lim_{t \to \infty} \left[ \log|x| \right]_1^t = \lim_{t \to \infty} \log|t|$$

So the improper integral converges for $p > 1$ and diverges for $p \leq 1$.

So now we can use the above to determine whether the following converge (without integrating anything):

1. $\int_1^\infty \frac{1}{\sqrt{x}} dx$ diverge.

2. $\int_1^\infty \frac{1}{x^{10}} dx$ converge.

3. $\int_1^\infty \frac{1}{x^{3/2}} dx$ converge.

4. $\int_1^\infty \frac{1}{x^{-1}} dx$ diverge.

(*) Mathematica demonstration.

Let's turn the above example into a theorem,

**Thm.** (p-test for infinite integrals) For any constant $\alpha > 0$, the integral

$$\int_{\alpha}^{\infty} \frac{1}{x^p}\,dx$$

converges for $p > 1$ and diverges for $p \leq 1$.

(*) mathematica demonstration.

**Thm.** (the eye test for infinite integrals) For an integral of the form,

$$\int_{\alpha}^{\infty} f(x)\,dx$$

if $\lim_{x \to \infty} f(x) \neq 0$ then the integral diverges.

BE CAREFUL! The contrapositive is not true, meaning,

$$\lim_{x\to 0} f(x) = 0 \not\Rightarrow \int_{\alpha}^{\infty} f(x)\,dx \text{ converges.}$$

think of the case of $f(x) = \frac{1}{x}$.

<u>Example 2.</u> $\int_0^{\infty} \sin x^2\,dx$

Let's look at the limit of the integrand as $x \to \infty$,

this function oscillates

$$\lim_{x\to\infty} \sin x^2 = \text{DNE}$$

$\neq 0 \Longrightarrow$ integral diverges by the eye test.

Now let's consider the integral $\int_1^{\infty} \frac{1}{x^3+x}\,dx$? We should guess that it converges, since it looks so much like $\int_1^{\infty} \frac{1}{x^3}dx$, but how do we prove this? We need more machinery than what we've stated above, so let's state the following theorem,

**Thm.** (Comparison test for improper integrals) If $f(x) \geq g(x) \geq 0$ on the interval $[\alpha, \infty)$, then

1. $\int_{\alpha}^{\infty} f(x)\,dx$ converges $\Longrightarrow \int_{\alpha}^{\infty} g(x)\,dx$ converges

If we know this

we can deduce this

2. $\int_a^\infty g(x)\,dx$ diverges $\Longrightarrow \int_a^\infty f(x)\,dx$ diverges

<u>Example 3.</u> Does the integral $\int_1^\infty \frac{1}{x^3+x}\,dx$ converge?

Let's use the comparison test, $f(x)=\frac{1}{x^3}$ and $g(x)=\frac{1}{x^3+x}$.

It is clear that $f(x) \geq g(x) \geq 0$ for all $x \geq 1$. Graph this if you have trouble seeing this.

Since $\int_1^\infty \frac{1}{x^3}\,dx$ converges $\underset{\text{by the comparison test.}}{\Longrightarrow}$ $\int_1^\infty \frac{1}{x^3+x}\,dx$ converges.