Lecture 9: notes

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From the warm up problem, we can pose the question,

Can we compute $\int_1^\infty \frac{1}{x^2}dx$?

The answer is yes but even if the integral doesn't exist we are able to show it with the following method,

Example 1. $\int_1^\infty \frac{1}{x^2}dx$.

We can't understand bounds of infinity so instead we write,

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{\alpha\to\infty} \underbrace{\int_1^t \frac{1}{x^2}dx}$$

Let's ignore the limit and just solve the integral with t as some constant.

$$\int_1^t \frac{1}{x^2}dx = \int_1^t x^{-2}dx = -\frac{1}{x}\Big]_1^t = -\frac{1}{t}+\frac{1}{1} = 1-\frac{1}{t}$$

Now put this back in the limit.

$$\lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1 \Rightarrow \boxed{\int_1^\infty \frac{1}{x^2}dx = 1}$$

We will call integrals like the above — where we are only concerned about limits to infinity (or negative infinity) — type I integrals.

The procedure for solving them is as follows:

1. For integrals of the type $\int_a^\infty f(x)\,dx$,
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$$\int_{\alpha}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{\alpha}^{t} f(x)\,dx$$

2. For integrals to negative infinity,

$$\int_{-\infty}^{\beta} f(x)\,dx = \lim_{t\to-\infty} \int_{t}^{\beta} f(x)\,dx$$

3. For integrals of the type,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\alpha} f(x)\,dx + \int_{\alpha}^{\infty} f(x)\,dx$$

where α is any constant of our choosing.

If the limit in the above exists, we say the integral converges. If the limit does not exist or goes off to an infinity, we say the integral diverges.

Example 2. $\displaystyle\int_{-\infty}^{0} \frac{dx}{x^2+1}$

$$\int_{-\infty}^{0} \frac{dx}{x^2+1} = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{x^2+1}$$

$$= \lim_{t\to-\infty} \Big[\arctan x\Big]_{t}^{0}$$

$$= \lim_{t\to-\infty} \left(\arctan 0 - \arctan t\right)$$

$$= \lim_{t\to-\infty} -\arctan t = \frac{-\pi}{2}$$

Example 3. $\displaystyle\int_{0}^{\infty} \frac{1}{x+1}\,dx$

$$\int_{0}^{\infty} \frac{1}{x+1}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{x+1}\,dx = \lim_{t\to\infty} \Big[\log(x+1)\Big]_{0}^{t}$$

$$= \lim_{t\to\infty} \left(\log(t+1) - \overset{0}{\cancel{\log 1}}\right) = \infty$$

$\Rightarrow$ $\boxed{\text{integral diverges}}$

Now let's consider a second type of improper integral with the following example,

Example 4. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

At first glance, this integral does not look like anything special, HOWEVER the integrand ($1/\sqrt{x}$) does not exist at $x=0$. So we write the following,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^1$$

$$= \lim_{t\to 0^+}\left(2-2\sqrt{t}\right) = 2.$$

Exercise. Compute the following if convergent, $\int_0^1 \frac{1}{x}\,dx$.

If the singularity appears in the middle of the bounds we must split the integral into two,

$$\int_0^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx$$

Each part must exist ON ITS OWN.

The integral only exist if both parts exist seperately. Consider the following example,

Example 3. $\int_0 \; x-3 \; dx$

The above integrand has a singularity at $x=3$,

$$\int_0^{10} \frac{1}{x-3}dx = \int_0^3 \frac{1}{x-3}dx + \int_3^{10} \frac{1}{x-3}dx$$

Let's consider this integral.

$$\int_0^3 \frac{1}{x-3}dx = \lim_{t\to 3^-}\int_0^t \frac{1}{x-3}dx$$

$$= \lim_{t\to 3^-}\Big[\log|x-3|\Big]_0^t = \lim_{t\to 3^-}\left(\log|t-3| - \log 3\right)$$

$= \infty \Rightarrow$ the original integral diverges.

We will call these integrals type 2 and their procedure is outlined below,

1. If $f(x)$ is discontinuous at $x=b$ then we write,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

or

$$\int_b^a f(x)dx = \lim_{t\to b^+}\int_t^a f(x)dx$$

2. If $a<c<b$ and $f(x)$ is discontinuous at $x=c$ then,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

solve these using (1).