# Lecture 6: notes

Thursday, June 27, 2019 9:42 AM

In this section we will make inverse substitutions using trig functions to solve a specific class of integrals.

usually, $u=f(x)$ is how we define substitution in this section we will instead define $x=f(u)$.

The following table summarizes the substitution we will use:

| Expression in integral | Substitution | Identity |
|---|---|---|
| $\sqrt{a^2-x^2}$ | $x=a\sin\theta$ | $1-\sin^2\theta=\cos^2\theta$ |
| $\sqrt{a^2+x^2}$ | $x=a\tan\theta$ | $1+\tan^2\theta=\sec^2\theta$ |
| $\sqrt{x^2-a^2}$ | $x=a\sec\theta$ | $\sec^2\theta-1=\tan^2\theta$ |

Based on the expression that appears in the integral, we make the approproapriate substitution and simplify using the corresponding identity.

Example 1. $\displaystyle\int\frac{dx}{\sqrt{1-x^2}}$

(*) some of you may recognize this as the antiderivative of arcsinx — which we will show

Use the substitution,

$u=\sin\theta$

$du=\cos\theta\, d\theta$

use identity

$$\int\frac{dx}{\sqrt{1-x^2}}=\int\frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}=\int\frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$$

$$=\int\frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$=\int d\theta=\theta$$

want to change back to $x$, $x=\sin\theta \Rightarrow \arcsin x=\theta$

$$\boxed{=\arcsin x}$$

$\arcsin x$

**Example 2.** $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$

from our "expressions"
$a^2 = 4 \rightarrow a = 2.$

Use the substitution,

$$x = 2\sec\theta$$
$$dx = 2\sec\theta\tan\theta\, d\theta$$

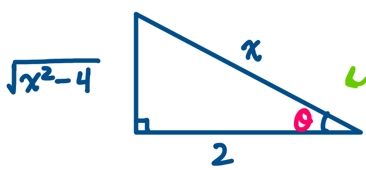
$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \int \frac{2\tan\theta\sec\theta\, d\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}} = \frac{1}{2}\int \frac{\tan\theta\cancel{\sec\theta}\, d\theta}{\sec^{\cancel{2}}\theta\sqrt{4(\sec^2\theta-1)}}$$

$$= \frac{1}{2}\int \frac{\cancel{\tan\theta}\, d\theta}{\sec\theta \cdot 2\cancel{\tan\theta}} = \frac{1}{4}\int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta$$

Now replacing $\theta$ is more challenging, so we need to draw a triangle,



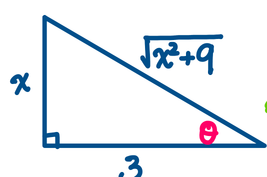

substitution tells us
$\sec\theta = \frac{x}{2} = \frac{\text{hypotenuse}}{\text{adjacent}}$

We can read $\sin\theta$ off this triangle (recall $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$),

$$\boxed{\sin\theta = \frac{\sqrt{x^2-4}}{x}}$$

**Example 3.** $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$

$x = 3\tan\theta$
$dx = 3\sec^2\theta\, d\theta$




it's good practice to draw the triangle every time we make the substitution

$$\int \frac{3\sec^2\theta\, d\theta}{\sqrt{9\tan^2\theta+9}} = \int \frac{\cancel{3}\sec^{\cancel{2}}\theta\, d\theta}{\cancel{3\sec\theta}}$$

$$= \int \sec\theta \, d\theta = \log|\sec\theta + \tan\theta|$$

$$= \log\left|\tfrac{1}{3}\sqrt{x^2+9} + \tfrac{1}{3}x\right|$$

use triangle!

If we see ANY quadratic polynomial we will want to use this procedure. That means we will have to complete the square if necessary.

<u>Example 4</u>. $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

Complete the square first,

$$3-2x-x^2 = 3-(x^2+2x+1)+1$$
$$3+1-(x+1)^2$$
$$= 4-(x+1)^2$$

Plug the above in the integral,

$$\int \frac{x\,dx}{\sqrt{4-(x+1)^2}} = \int \frac{(u-1)\,du}{\sqrt{4-u^2}} = \int \frac{u}{\sqrt{4-u^2}}\,du - \int \frac{1}{\sqrt{4-u^2}}\,du$$

use substitution,
$u=x+1$, $du=dx$
$u-1=x$

<u>Exercise</u>. Solve the above integrals.