Lecture 5: note

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In the following we will address integrals that are of the form,

$$\boxed{\int \sin^n x \cos^m x \, dx} \quad \text{or} \quad \boxed{\int \tan^n x \sec^m x \, dx}$$

where m & n are non-negative integers. In these integrals we will usually need to use trigonometric identities BEFORE making a u-substitution.

which we should be comfortable with!

Summary: $\int \cos^n x \sin^m x \, dx$

1) if either n or m is odd, use identity $\sin^2 x + \cos^2 x = 1$ and make a u-sub.
2) If both m & n are even, use half angle identities,
   $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, $\cos^2 x = \frac{1}{2}(1+\cos(2x))$

Example 1. $\int \sin^5 x \, dx$

Since the power of sine is ODD we strip off one $\sin x$ and convert the others using $\sin^2 x = 1 - \cos^2 x$.

$$\int \sin^4 x \cdot \sin x \, dx = \int (\sin^2 x)^2 \cdot \sin x \, dx$$

left behind for u-sub

NO POWERS!

$u = \cos x$

$du = -\sin x \, dx$

$$= \int (1-\cos^2 x)^2 \cdot \sin x \, dx$$

$$= \int (1-u^2)^2 \cdot (-du)$$

$$= -\int (1-2u^2+u^4)\, du = -u + \frac{2}{3}u^3 + \frac{1}{5}u^5$$

$$\boxed{= -\cos x + \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x}$$

Example 2. $\int \cos^3 x \sin^2 x \, dx$

Since the exponent on $\cos x$ is odd we strip one off and replace the other two.

$$\int \cos^3 x \sin^2 x \, dx = \int \cos x \cdot \cos^2 x \sin^2 x \, dx$$

$$= \int \cos x \cdot (1-\sin^2 x) \sin^2 x \, dx$$

$u = \sin x$
$du = \cos x \, dx$

$$= \int (1-u^2) u^2 \, du = \int (u^2 - u^4) \, du$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 = \boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x \, .}$$

Example 3. $\int \cos^2 x \sin^2 x \, dx$

In this case both exponents are even so we have to use the double angle formulas,

$$\int \sin^2 x \cdot \cos^2 x \, dx = \int \frac{1}{2}(1-\cos 2x) \cdot \frac{1}{2}(1+\cos 2x) \, dx$$

$$= \frac{1}{4} \int (1 - \cos^2 2x) \, dx$$

still can't handle this because of exponent.

$$= \frac{1}{4} \int \left[1 - \frac{1}{2}(1+\cos 4x) \right] dx$$

$$= \frac{1}{4} \int 1 \, dx - \frac{1}{8} \int 1 \, dx - \frac{1}{8} \int \cos 4x \, dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8} \cdot \frac{1}{4} \sin 4x = \boxed{\frac{1}{8}x - \frac{1}{32}\sin 4x}$$

Try the following on your own,

1) $\int \cos^4 x \cdot \sin^3 x \, dx$

2) $\int \cos^4 x \, dx$

3) $\int \cos^5 x \sin^3 x \, dx$.

Summary: $\int \tan^n x \sec^m x \, dx$

1) if the power of secant is even, leave a $\sec^2 x$ and replace the rest using

$\sec^2 x = 1 + \tan^2 x$.

2) if the power of tangent is odd and there's at least one secant, save $\tan x \sec x$ and replace the rest of the tangents using $\tan^2 x = \sec^2 x - 1$.

The above is obviously not all possible scenarios, and the rest we will have to tackle as they come.

Example 4. $\int \sec^6 x \, dx$

Power of secant is even,

$$\int \sec^6 x \, dx = \int \sec^4 x \cdot \sec^2 x \, dx$$

$$= \int (1+\tan^2 x)^2 \sec^2 x \, dx \qquad u = \tan x, \quad du = \sec^2 x \, dx$$

$$= \int (1+u^2)^2 \, du$$

$$= \int (1+2u^2+u^4) \, du = u + \frac{2}{3}u^3 + \frac{1}{5}u^5$$

$$\boxed{= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x.}$$

Example 5. $\int \tan^5 x \sec^3 x \, dx$

The power of tan is odd so,

$$\int \tan^4 x \cdot \sec^2 x \cdot \tan x \sec x \, dx = \int (\sec^2 x - 1)^2 \sec^2 x \cdot \tan x \sec x \, dx \qquad u = \sec x, \quad du = \tan x \sec x \, dx$$

$$= \int (u^2-1)^2 u^2 \cdot du$$

$$= \int (u^6 - 2u^4 + u^2) \, du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3$$

$$\boxed{= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x.}$$

Example 6. $\int \sec^3 x \, dx$

This doesn't fit into either of the criteria, so we have to improvise,

$u = \sec x \qquad du = \tan x \sec x \, dx$

$dv = \sec^2 x \, dx \qquad v = \tan x$

replace this

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

If you've never solved this before, it is a good exercise. we will take the ans for granted

$$+ \int \sec^3 x \, dx \qquad\qquad + \int \sec^3 x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \log|\sec x + \tan x|$$

$$\int \sec^3 x \, dx = \boxed{\frac{1}{2} \sec x \tan x + \frac{1}{2} \log|\sec x + \tan x|}$$

Try the following on your own,

1) $\int \tan^4 x \sec^4 x \, dx$

2) $\int \tan^4 x \sec x \, dx$