Lecture 4: notes

Monday, June 24, 2019 1:27 PM

So far we've covered decomposing rational functions with denominators that factor into linear factors (distinct or not).
Today we want to cover irreducible quadratic factors and integrating what we have decomposed.

Example 1. Decompose $\dfrac{2x^2-x+4}{x^3+4x}$

As always we should factor the denominator as much as possible,
$Q(x) = x^3+4x = x(x^2+4)$

can't factor anymore!

So how do we decompose the rational function when we have an irreducible quadratic?

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

put a linear term here

same as before.

Now the procedure is the same as yesterday!

Exercise. Solve for A, B, and C above.

Case III: Irreducible, distinct quadratic factors.

If $Q(x)$ has a distinct, irreducible quadratic factor ax^2+bx+c then we use the factor,

$$\frac{Ax+B}{ax^2+bx+c}$$

in decomposition.

Example 2. Decompose $\dfrac{x^2-29x+5}{(x-4)^2(x^2+3)}$

$Q(x)$ is already factored above, so let's decompose,

$$\frac{x^2-29x+5}{(x-4)^2(x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$$

$$\begin{aligned} x^2-29x+5 &= A(x-4)(x^2+3) + B(x^2+3) + (Cx+D)(x-4)^2 \\ &= A(x^3-4x^2+3x-12) + B(x^2+3) + (Cx+D)(x^2-8x+16) \end{aligned}$$

$$\left.\begin{aligned} x^3 &: \; 0 = A + C \\ x^2 &: \; 1 = -4A + B + D - 8C \\ x^1 &: \; -29 = 3A + 16C - 8D \\ x^0 &: \; 5 = -12A + 3B + 16D \end{aligned}\right\} \; A=1,\ B=-5,\ C=-1,\ D=2.$$

Exercise. Write out the decomposition of $f(x) = \dfrac{x^2+4x-10}{(x+3)(x-1)^3(x^2+2)}$ without solving for the unknown coefficients.

Case IV: Repeated, irreducible quadratic terms.

If $Q(x)$ has a repeated irreducible factor $(ax^2+bx+c)^r$ we use the terms,

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

we could extend this pattern of procedures to any irreducible polynomial repeated any number of times. We will stop here though.

Example 3. Write how to decompose $f(x) = \dfrac{5x^3-20x+2}{(x^4-4x^2)(x^2+1)^2(x-2)}$

Just because $Q(x)$ is written with parenthases, doesn't mean we are done factoring it,

$$Q(x) = (x^4-4x^2)(x^2+1)^2(x-2)$$

$$= x^2(x^2-4)(x^2+1)^2(x-2)$$

$$= x^2(x^2-4)(x^2+1)^2(x-2)$$

$$= x^2(x-2)(x+2)(x^2+1)^2(x-2)$$

$$= x^2(x-2)^2(x+2)(x^2+1)^2$$

$$f(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{x+2} + \frac{Fx+G}{x^2+1} + \frac{Hx+I}{(x^2+1)^2}$$

Glad we don't have to solve that!

<u>Example 4.</u> $\displaystyle\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,dx$

Let's decompose first:

$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$x^3+10x^2+3x+36 = A(x^2+4)^2 + (Bx+C)(x-1)(x^2+4) + (Dx+E)(x-1)$$

$$= A(x^4+8x^2+16) + (Bx+C)(x^3-x^2+4x-4) + (Dx+E)(x-1)$$

$$\left.\begin{aligned} x^4 &: 0 = A+B \\ x^3 &: 1 = C-B \\ x^2 &: 10 = 8A+4B-C+D \\ x^1 &: 3 = -4B+4C-D+E \\ x^0 &: 36 = 16A-4C-E \end{aligned}\right\} \Rightarrow A=2,\ B=-2,\ C=-1,\ D=1,\ E=0.$$

Now let's integrate,

$$\int \left(\frac{2}{x-1} + \frac{-2x-1}{x^2+4} + \frac{x}{(x^2+4)^2} \right) dx$$

$$= \int \frac{2}{x-1}\,dx - \underbrace{\int \frac{2x+1}{x^2+4}\,dx}_{} + \int \frac{x}{(x^2+4)^2}\,dx$$

u-sub $u=x-1$

u-sub $u=x^2+4$

$$= \int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

u-sub $u=x^2+4$

u-sub $u=\frac{x}{2}$

u-sub $u = x + 9$

this is arctan
anti-derivative.