Lecture 3: notes

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Goal: to formalize decomposing fractions into the smallest pieces possible.

We are considering rational functions, that is functions of the form,

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. Moreover, we require $f(x)$ to be proper meaning $\deg Q(x) > \deg P(x)$.

The degree of a polynomial is the largest exponent,

- $\deg(x^2+1) = 2$
- $\deg(x^4 - x) = 4$
- $\deg(x^3+3x-2) = 3$
- $\deg(3x-1) = 1$
- $\deg(2) = 0$
- $\deg(-4x^5+x^3) = 5$

$\deg P(x) \geq \deg Q(x)$

If a rational function is improper, we must first long divide like the following,

<u>Example 1</u>. Make the following proper, $\dfrac{x^2+3x-1}{x^2-x+2}$.

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2-x+2 & x^2+3x-1 \\ & -(x^2-x+2) \\ \hline & \quad 4x-3 \end{array}$$

degree < 2 so we are done.

$$\frac{x^2+3x-1}{x^2-x+2} = 1 + \frac{4x-3}{x^2-x+2}$$

Example 2. Simplify $\dfrac{x^3+3x^2-x-1}{x+4}$ by long dividing

$$
\begin{array}{r l}
 & x^2 - x + 3 \\
x+4 \,\big)\!\! & \overline{x^3+3x^2-x-1} \\
 & -(x^3+4x^2) \\
 & \quad -x^2-x-1 \\
 & \quad -(-x^2-4x) \\
 & \qquad 3x-1 \\
 & \qquad -(3x+12) \\
 & \qquad\quad 13
\end{array}
$$

So we have, $\dfrac{x^3+3x^2-x-1}{x+4} = x^2-x+3+\dfrac{13}{x+4}$.

Now let's assume all rational functions we deal with are proper.

Given a rational function $f(x) = P(x)/Q(x)$ the first step is to factor $Q(x)$ as much as possible. How we decompose depends on how $Q(x)$ factors.

<u>Case I</u>: $Q(x)$ is a product of distinct, linear factors.

no repeating factors

factors look like $ax+b$ (for constants a & b).

Let's see some examples of $Q(x)$ that fit this criteria:

- $Q(x) = x^2+3x+2$
- $Q(x) = (x-2)(x+3)(2x-1)$

· $Q(x) = 4x^3 + 12x^2 - x - 3$

Exercise. Factor the last example.

Generally, if $Q(x)$ factors as $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$, then there exists constants $A_1, \dots, A_n$ such that,

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

How do we find these constants? We will re-add the right hand side and create a linear system from the numerators.

Example 3. $f(x) = \dfrac{2x+5}{x^2+5x+6}$

$Q(x) = x^2+5x+6 = (x+3)(x+2)$ linear ☑ distinct ☑

$$\frac{A_1}{x+3} + \frac{A_2}{x+2} = \frac{A_1(x+2)+A_2(x+3)}{(x+3)(x+2)} = \frac{2x+5}{(x+3)(x+2)}$$

$$A_1(x+2) + A_2(x+3) = 2x+5$$
$$A_1x + A_1 \cdot 2 + A_2x + A_2 \cdot 3 = 2x+5$$

Now since A_1 & A_2 are constants, we can group terms by their powers of x to get a linear system,

constant terms (x^0): $2A_1 + 3A_2 = 5$

$x^1 = x$ terms: $A_1\cancel{x} + A_2\cancel{x} = 2\cancel{x}$ x's cancel.

now we see a linear system of 2 variables with 2 equations.

Exercise. Solve the above system for A_1 & A_2.

Example 4. $f(x) = \dfrac{x^2+3x}{(x+1)(x+2)(x-1)}$

$$\frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{x-1} = \frac{x^2+3x}{(x+1)(x+2)(x-1)}$$

$$\Downarrow$$

$$A_1(x+2)(x-1) + A_2(x+1)(x-1) + A_3(x+1)(x+2) = x^2+3x$$

$$A_1(x^2+x-2) + A_2(x^2-1) + A_3(x^2+3x+2) = x^2+3x$$

x^0 terms: $-2A_1 - A_2 + 2A_3 = 0$ ← no constant terms on the RHS.

x^1 terms: $A_1\cancel{x} + 3A_3\cancel{x} = 3\cancel{x}$

x^2 terms: $A_1\cancel{x^2} + A_2\cancel{x^2} + A_3\cancel{x^2} = \cancel{x^2}\,1$

Now let's solve the system,

$$\left.\begin{aligned} -2A_1 - A_2 + 2A_3 &= 0 \\ A_1 + A_2 + A_3 &= 1 \end{aligned}\right\} \text{add these.}$$

$$\left.\begin{aligned} -A_1 + 3A_3 &= 1 \\ A_1 + 3A_3 &= 3 \end{aligned}\right\} \text{add these.}$$

$$6A_3 = 4 \Rightarrow \boxed{A_3 = \frac{2}{3}}$$

$$A_1 + 3A_3 = A_1 + 3\left(\frac{2}{3}\right) = 3$$

$$A_1 + 2 = 3 \Rightarrow \boxed{A_1 = 1}$$

$$A_1 + A_2 + A_3 = 1 + A_2 + \frac{2}{3} = 1$$

$$\frac{5}{3} + A_2 = 1 \Rightarrow \boxed{A_2 = -\frac{2}{3}}$$

So now we need to address the situation where factors in $Q(x)$ are repeated and the situation where irreducible factors have degree more than 1 (i.e. x^2+4, x^2+x+6).

Case II: $Q(x)$ is a product of linear factors, some of which are repeated.

repeated.

The procedure is mostly the same as the above we just need to know how to deal with the repeated factors. Consider the following example.

Example 5. $f(x) = \dfrac{4x}{x^3 - x^2 - x + 1}$

Let's factor $Q(x)$,

$$
\begin{aligned}
Q(x) = x^3 - x^2 - x + 1 &= x^2(x-1) - (x-1) \\
&= (x^2-1)(x-1) \quad \text{not done!} \\
&= (x-1)(x+1)(x-1) \\
&= (x-1)^2(x+1) \quad \begin{array}{l} \text{linear } \checkmark \\ \text{distinct } \times \end{array}
\end{aligned}
$$

Here is our blueprint for decomposing,

$$
\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}
$$

$$
= \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}
$$

Now just focus on the numerators,

$$
4x = A(x^2-1) + B(x+1) + C(x^2-2x+1)
$$
$$
4x = (A+C)x^2 + (B-2C)x + (-A+B+C)
$$

Write 3 equations with 3 variables,

$$
\begin{aligned}
0 &= -A + B + C \\
4 &= B - 2C \Rightarrow B = 4 + 2C \\
0 &= A + C \Rightarrow A = -C
\end{aligned}
$$

$$
0 = C + 4 + 2C + C = 4 + 4C
$$

$-4 = 4C \Rightarrow C = -1.$

$0 = A - 1 \Rightarrow A = 1$

$0 = -1 + B - 1 = -2 + B \Rightarrow B = 2.$

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times, then instead of the single term $A_1/(a_1x+b_1)$ we have the r terms,

$$\frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r}$$

That's all we will cover today.