Lecture 2: notes

Tuesday, May 21, 2019 11:22 AM

Motivation via product rule.

Let's recall the product rule of differentiation via a couple of examples,

Example 1. $F(x) = xe^x$

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}(x)\cdot e^x + x\cdot\frac{d}{dx}(e^x)$$

$$= 1\cdot e^x + xe^x$$

I'll always be referring to the natural log unless stated otherwise.

Example 2. $G(x) = x\log x$

$$\frac{d}{dx}(G(x)) = \frac{d}{dx}(x)\cdot\log x + x\cdot\frac{d}{dx}(\log x)$$

$$= 1\cdot\log x + x\cdot\frac{1}{x}$$

$$= \log x + 1$$

Now let's integrate both sides of the equation in the above examples,

Mathematically this means $\left[\underbrace{r(x) = s(x)}_{\Updownarrow}\right] \Rightarrow \left[\underbrace{\int r(x)\,dx = \int s(x)\,dx}_{\Updownarrow}\right]$.

if two functions are identical $\Rightarrow$ the area under the functions is the same

Example 1' $\frac{d}{dx}(F(x)) = e^x + xe^x$

$$\int \frac{d}{\cancel{dx}}(F(x))\,\cancel{dx} = \int (e^x + xe^x)\,dx$$

$$\int d(F(x)) = \int e^x dx + \int x e^x dx$$

this may look odd, but think of how $\int dx = x$

$$F(x) = \int e^x dx + \int x e^x dx$$

$$x e^x = \int e^x dx + \int x e^x dx$$

no integral here! — we know how to solve this — Not sure how to solve this!

The above is therefore a prescription for solving $\int x e^x dx$.

<u>Example 2</u>: $\frac{d}{dx}(G(x)) = \log x + 1$

$$\int \frac{d}{\cancel{dx}}(G(x))\,\cancel{dx} = \int \log x \, dx + \int 1 \, dx$$

$$\int d(G(x))$$

$$G(x) = \underbrace{\int \log x \, dx} + \int dx$$

In this example we have a prescription for solving for $\int \log x \, dx$.

Let's generalize the above examples to any function of the form $y = f(x) \cdot g(x)$.

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x))$$

$$\int \frac{d}{\cancel{dx}}\left(f(x) \cdot g(x)\right) \cancel{dx} = \int \frac{d}{dx}(f(x)) \cdot g(x)\, dx + \int f(x) \cdot \frac{d}{dx}(g(x))\, dx$$

DONT ~~CANCEL~~!

$$\boxed{f(x)\cdot g(x) = \int \frac{d}{dx}(f(x))\cdot g(x)\,dx + \int f(x)\cdot \frac{d}{dx}(g(x))\,dx}$$

Hope that we know how to solve one of these.

Integration by parts formalism.

Let's reformulate the above process so it's more tailored to suiting our need — solving integrals.

Def'n. Suppose we have an integral of the form,

$$\int f(x)\cdot g'(x)\,dx$$

then the integration by parts formula allows us to write,

$$\underbrace{\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx}_{\text{This is typically written in shorthand as,}}$$

$$\boxed{\int u\cdot dv = u\cdot v - \int v\cdot du}$$

$u = f(x)$
$v = g(x)$

When considering a definite integral the above formula becomes

$$\boxed{\int_a^b u\,dv = u\cdot v\Big|_a^b - \int_a^b v\,du}$$

notation for "evaluate from",
i.e. $f(x)\Big|_a^b = f(a) - f(b)$.

Let's start with an example before the general procedure,

<u>Example 3.</u> $\int x e^x\,dx$

$u = x \qquad dv = e^x dx$
$du = dx \qquad v = e^x$

$$\int x e^x dx = x e^x - \int e^x dx$$
$$= x e^x - e^x$$

So when doing integration by parts consider the following steps:

1. Consider the possible u & dv combinations, pick the one you think is right.
2. Use the formula
3. Analyze the new integral formed by considering the following questions:
   i) Can I solve the new integral?
   ii) Should I start over?
   iii) Can I use integration by parts again?

<u>Example 4.</u> $\int \underbrace{x\cos x}\,dx$

Well I can integrate x & $\cos x$
but not $x\cos x$, so I clearly want
to eliminate one.

$u = x$ $\quad dv = \cos x\, dx$

$du = dx$ $\quad v = \sin x$

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx$$
$$= x\sin x + \cos x$$

<u>Example 5.</u> $\int x^2 e^x\, dx$

I can't solve this immediately,
but I know I can if I use
IBP again.

$u = x^2$ $\quad dv = e^x dx$

$du = 2x\, dx$ $\quad v = e^x$

$$\int x^2 e^x\, dx = x^2 e^x - \int 2x e^x\, dx$$
$$= x^2 e^x - 2\left(x e^x - e^x\right)$$

not always
clear what is
easier though.

(*) if the new integral is "easier" to solve than the old one
you probably shouldn't start over