Concepts

Euler circuit  Euler path

Problems

1. For which \( n \) (for part (d), for which \( m \) and \( n \)) do the following graphs have an Euler circuit? An Euler path but not an Euler circuit?

(a) \( K_n \)

Solution. The degree of any vertex in \( K_n \) is \( n - 1 \), so \( K_n \) has an Euler circuit only for \( n \) odd. For \( n \) even, it has an Euler path for \( n = 2 \), but not for any higher \( n \).

(b) \( C_n \)

Solution. \( C_n \) has an Euler circuit for all \( n \).

(c) \( W_n \)

Solution. \( W_n \) always has \( n \) vertices of degree 3. Since \( W_n \) only makes sense for \( n \geq 3 \), this means \( W_n \) never has an Euler path or Euler circuit.

(d) \( K_{m,n} \), the graph on vertex set \( V_1 \cup V_2 \), where \( |V_1| = m \) and \( |V_2| = n \), such that \( \{u, v\} \) is an edge if and only if \( v \in V_1 \) and \( u \in V_2 \).

Solution. The degree of a vertex in \( V_1 \) is \( n \) and the degree of a vertex in \( V_2 \) is \( m \). If \( m \) and \( n \) are both even, then there is an Euler circuit; otherwise, there is not. If one of \( m \) or \( n \) (say \( n \)) is odd and the other (\( m \)) is 2, there is an Euler path, since the two vertices of \( V_1 \) will have odd degree and the vertices of \( V_2 \) will have degree 2. Any other choice would result in too many vertices of odd degree.

2. Construct an Euler circuit if one exists. If no Euler circuit exists, construct an Euler path if one exists.

Solution. The top graph has an Euler circuit: start at \( v_1 \), following the pink circuit. Then follow the blue circuit from \( v_1 \) to \( v_2 \), then follow the orange circuit from \( v_2 \) back to \( v_2 \) and follow the rest of the blue circuit back to \( v_1 \).

The middle graph has no Euler circuit, but does have an Euler path from \( u \) to \( v \). Start at \( u \), follow the orange circuit back to \( u \), then follow the blue path to \( v \).

The bottom graph doesn’t have an Euler circuit (it has vertices of odd degree), but does have an Euler path visiting the vertices in the following order: \( uxyuvxyuv \).