Concepts

graph	degree	cycle
vertex	adjacency matrix	wheel
edge	complete graph	bipartite

Problems

1. Let $G = (V, E)$ be a simple (that is, without loops or multiple edges) undirected graph, and let R be a relation on V where uRv if and only if $\{u, v\}$ is an edge of G. What properties does R have?

Solution. R is symmetric, since an edge $\{u, v\}$ of G means that (u, v) and (v, u) are in R. It is also irreflexive, meaning that for all $v \in V$, $(v, v) \notin R$, since G has no loops.

2. Let G be a graph with v vertices and e edges. Let M be the maximum degree of any vertex and m the minimum degree. Show that $m \leq 2e/v \leq M$

Solution. By the handshake lemma,

$$2e = \sum_{u \in V} d(u).$$

We also know that $m \leq d(u) \leq M$ for all $u \in V$, so if we sum over all $u \in V$, we get $vm \leq 2e \leq vM$. Dividing through by v gives the desired inequalities.

3. For which values of n are the following graphs bipartite?

(a) C_n

Solution. If n is even, C_n is bipartite; otherwise, it's not.

(b) K_n

Solution. For $n = 0, 1, 2$. Otherwise, K_n contains C_3 as a subgraph, and C_3 is not bipartite.

(c) W_n

Solution. For $n = 1$. Otherwise, W_n contains C_3 as a subgraph, which is not bipartite.

4. A *proper k-coloring* of a graph G is an assignment of k colors to the vertices of G so that no adjacent vertices are the same color. For the following graphs, what is the smallest k such that there exists a proper k-coloring of G?

(a) C_n

Solution. For even n, C_n is bipartite, so there is a proper 2-coloring of C_n (and there is no proper 1-coloring, since C_n has at least one edge).

For odd n, C_n is not bipartite, so there does not exist a proper 2-coloring. There is always a proper 3-coloring (make one vertex color 1, then alternate between colors 2 and 3).

(b) K_n

Solution. Since there is an edge between any 2 vertices in K_n, every vertex must be a different color in a proper coloring. So there is a proper n-coloring and no proper $(n-1)$-colorings.

(c) W_n

Solution. If $n = 1$, W_n is bipartite. $W_2 = K_3$, which has a proper 3-coloring and no proper 2-colorings. For $n \geq 3$, W_n is not bipartite, so there are no proper 2-colorings. For even $n \geq 3$, W_n has a proper 3-coloring (color the cycle properly with 2 colors and make the central vertex the 3rd color). For odd $n \geq 3$, W_n doesn't have any proper 3-colorings (in any 3-coloring, the cycle must use all three colors, so there's no options left for the central vertex), but it does have a proper 4-coloring.