Concepts

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Problems

1. Let $G = (V, E)$ be a simple (that is, without loops or multiple edges) undirected graph, and let $R$ be a relation on $V$ where $uRv$ if and only if $\{u, v\}$ is an edge of $G$. What properties does $R$ have?

Solution. $R$ is symmetric, since an edge $\{u, v\}$ of $G$ means that $(u, v)$ and $(v, u)$ are in $R$. It is also irreflexive, meaning that for all $v \in V$, $(v, v) \notin R$, since $G$ has no loops.

2. Let $G$ be a graph with $v$ vertices and $e$ edges. Let $M$ be the maximum degree of any vertex and $m$ the minimum degree. Show that $m \leq 2e/v \leq M$

Solution. By the handshake lemma,

$$2e = \sum_{u \in V} d(u).$$

We also know that $m \leq d(u) \leq M$ for all $u \in V$, so if we sum over all $u \in V$, we get $vm \leq 2e \leq vM$. Dividing through by $v$ gives the desired inequalities.

3. For which values of $n$ are the following graphs bipartite?

(a) $C_n$

Solution. If $n$ is even, $C_n$ is bipartite; otherwise, it’s not.

(b) $K_n$

Solution. For $n = 0, 1, 2$. Otherwise, $K_n$ contains $C_3$ as a subgraph, and $C_3$ is not bipartite.

(c) $W_n$

Solution. For $n = 1$. Otherwise, $W_n$ contains $C_3$ as a subgraph, which is not bipartite.

4. A proper $k$-coloring of a graph $G$ is an assignment of $k$ colors to the vertices of $G$ so that no adjacent vertices are the same color. For the following graphs, what is the smallest $k$ such that there exists a proper $k$-coloring of $G$?

(a) $C_n$

Solution. For even $n$, $C_n$ is bipartite, so there is a proper 2-coloring of $C_n$ (and there is no proper 1-coloring, since $C_n$ has at least one edge).

For odd $n$, $C_n$ is not bipartite, so there does not exist a proper 2-coloring. There is always a proper 3-coloring (make one vertex color 1, then alternate between colors 2 and 3).
(b) $K_n$

*Solution.* Since there is an edge between any 2 vertices in $K_n$, every vertex must be a different color in a proper coloring. So there is a proper $n$-coloring and no proper $(n - 1)$-colorings.

(c) $W_n$

*Solution.* If $n = 1$, $W_n$ is bipartite. $W_2 = K_3$, which has a proper 3-coloring and no proper 2-colorings. For $n \geq 3$, $W_n$ is not bipartite, so there are no proper 2-colorings. For even $n \geq 3$, $W_n$ has a proper 3-coloring (color the cycle properly with 2 colors and make the central vertex the 3rd color). For odd $n \geq 3$, $W_n$ doesn’t have any proper 3-colorings (in any 3-coloring, the cycle must use all three colors, so there’s no options left for the central vertex), but it does have a proper 4-coloring.