Concepts

P-closure	transitive closure	equivalence relation
reflexive closure	path of length n	equivalence class
symmetric closure	connectivity relation	partition

Problems

1. Let R be a relation on A, and D the digraph of R. How should you chance D to get the digraph of the reflexive closure of R? What about the symmetric closure? The transitive closure?

Solution. To get the digraph of the reflexive closure, add a loop at every vertex. To get the digraph of the symmetric closure, for every arrow $i \to j$, add an arrow $j \to i$. For the transitive closure, for every path $a_0 \to a_1 \to \cdots \to a_n$, add an arrow $a_0 \to a_n$. Alternately, for every path $a_0 \to a_1 \to a_2$, add an arrow $a_0 \to a_2$ and then repeat this process with the resulting digraph until no new arrows result.

2. Let R be the relation $\{(a,b)|a < b\}$ on $\mathbb{Z}$. What is the P-closure of R where P is

(a) reflexive?

Solution. The reflexive closure of R is "less than or equal to", $\{(a,b)|a \leq b\}$.

(b) symmetric?

Solution. The symmetric closure of R is "not equal to", $\mathbb{Z}^2 - \{(a,a)|a \in \mathbb{Z}\}$

(c) transitive?

Solution. R is transitive, so it is its own transitive closure.

3. Give an example of a property P such that if R has property P, so does R^*.

Solution. Reflexivity and symmetry are both properties like this. Reflexivity is perhaps easier to see: say R is a relation on A. $R \subseteq R^*$, so if $(a,a) \in R$ for all $a \in A$, then $(a,a) \in R^*$ for all $a \in A$. The reason symmetry is preserved is that in a symmetric relation, if $a_0 \to a_1 \to \cdots \to a_n$ is a path in R, so is $a_0 \leftarrow a_1 \leftarrow \cdots \leftarrow a_n$, so R^n is symmetric for all n. This implies R^* is also symmetric.

4. Let A be a nonempty set, and f a function with domain A. Show that $R = \{(a,b)|f(a) = f(b)\}$ is an equivalence relation on A. What are the equivalence classes of this relation?

Solution. Reflexivity: $f(a) = f(a)$, so aRa.
Symmetry: If $f(a) = f(b)$, then $f(b) = f(a)$, so aRb implies bRa.
Transitivity: If $f(a) = f(b)$ and $f(b) = f(c)$, then $f(a) = f(c)$, so aRb and bRc imply aRc.

Let $a \in A$. $[a] = \{b \in A|f(b) = f(a)\}$; in words, the equivalence classes group the elements of A according to where they are mapped by f, which we can write in a more familiar way. Let $f(a) = z$. Then $[a] = f^{-1}(\{z\})$.

5. List the ordered pairs in the equivalence relations produced by the following partitions of $\{1, 2, 3, 4, 5, 6\}$.

(a) $\{1\}, \{2, 4, 6\}, \{3, 5\}$

Solution. The corresponding equivalence relation is $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 4), (4, 6), (2, 6), (4, 2), ($

(b) $\{1, 4\}, \{2, 5\}, \{3, 6\}$

Solution. $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (2, 5), (5, 2), (3, 6), (6, 3)\}$.

6. Let R be the relation on $\mathbb{Z}^2$ such that $(a, b)R(c, d)$ iff $ac = bd$. By your homework this week, R is an equivalence relation. What are the equivalence classes of $(1, 2)$ and $(3, 1)$? The set of all equivalence classes is a familiar set- what is it? (Hint: think about the ratio a/b.)

Solution. $[(1, 2)] = \{(a, 2a)|a \in \mathbb{Z}\}$, and $[(3, 1)] = \{(3a, a)|a \in \mathbb{Z}\}$.

To answer the other question, notice that if $(c, d) \in [(a, b)]$ and d and b are nonzero, $c/d = a/b$ (if either d or b is zero, then $[(a, b)] = [(0, 0)]$). In fact, this characterizes the elements of $[(a, b)]$ as long as $[(a, b)] \neq [(0, 0)]$. So equivalence classes correspond exactly to elements of $\mathbb{Q}$; each element of the equivalence class gives a different way to write a particular rational number as a ratio of two integers (with the exception of elements of $[(0, 0)]$, which is a little weird).