**Concepts**

<table>
<thead>
<tr>
<th>permutation</th>
<th>binomial coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>r-permutation</td>
<td>combinatorial proof</td>
</tr>
<tr>
<td>r-combination</td>
<td></td>
</tr>
</tbody>
</table>

**Problem Solutions**

1. Let $X$ be the set of words of length $n$ in letters $a, b, c, d$.

(a) How many elements of $X$ contain exactly $4$ $a$’s?

*Solution.* To count this, we count the number of ways the $a$’s can be arranged first and then count the number of ways to fill in the rest of the word. If $n < 4$, there are no elements of $X$ with exactly $4$ $a$’s, so we assume $n \geq 4$. To place the $a$’s, we choose $4$ positions in the word-order doesn’t matter, so the number of ways to do this is $\binom{n}{4}$. Then there are $n - 4$ positions left, which we can fill with $b, c, d$ as we wish- there are $3^{n-4}$ ways to do this. By the product rule, there are $3^{n-4} \binom{n}{4}$ words in $X$ with exactly $4$ $a$’s.

(b) How many elements of $X$ contain exactly $k$ $a$’s, for $0 \leq k \leq n$?

*Solution.* We do exactly the same thing as above, except with $k$ in place of $4$. In particular, if $k > n$, this number is $0$. Otherwise, there are $3^{n-k} \binom{n}{k}$ words in $X$ with exactly $k$ $a$’s.

(c) How many elements of $X$ contain at least $k$ $a$’s, for $0 \leq k \leq n$?

*Solution.* Again, if $k > n$, this number is $0$. Otherwise, the elements containing at least $k$ $a$’s are those that contain $k$ $a$’s, $k + 1$ $a$’s, $\ldots$, or $n$ $a$’s. So to count them, we count how many elements there are containing exactly $j$ $a$’s for $k \leq j \leq n$ and then add these numbers. So by part (b), there are

$$
\sum_{j=k}^{n} 3^{n-j} \binom{n}{j}
$$

of these elements.

2. How many ways are there to seat people $p_1, \ldots, p_k$ at a circular table with $n$ seats numbered $1, \ldots, n$ clockwise around the circle? How many ways are there to seat people $p_1, \ldots, p_k$ at a circular table with $n$ seats if 2 seating arrangements are considered the same if they can be obtained from one another by rotating the table? (This is essentially the same as having unnumbered seats.)

*Solution.* For the first question, if the seats are numbered, seating people $p_1, \ldots, p_k$ in $n$ seats is just a $k$-permutation of $n$, so there are $n!/ (n-k)!$ ways to do this.

For the second question: it’s easiest to count this by figuring out how many seating arrangements with numbered seats we can get out of one unnumbered seating arrangement. Given an unnumbered seating arrangement, as soon as we choose with seat is seat $1$, then we’ve determined the numbers of the rest of the seats. There are $n$ choices for seat $1$, and each one gives a different numbered
seating arrangement. So the number of numbered seating arrangements is \( n \) times the number of unnumbered seating arrangements, which means there are \( n!/(n(n-r)! \) unnumbered seating arrangements.

3. How many terms are there in the expansion of \((2x - 1)^{1/2}\), after like terms have been collected? What is the coefficient of \(x^4\) in this expansion?

Solution. By the Binomial Theorem,

\[
(2x - 1)^{1/2} = \sum_{k=0}^{\lfloor \frac{12}{2} \rfloor} \binom{12}{k} (-1)^{n-k} (2x)^k
\]

so the expansion has 13 terms. The coefficient of \(x^4\) is \(\binom{12}{4} (-1)^8 (2)^4 = 16 \binom{12}{4}\).

4. Give 2 proofs (one combinatorial and one algebraic) that if \( n \in \mathbb{Z}^+ \), then

\[
\binom{2n}{2} = 2 \binom{n}{2} + n^2.
\]

Solution. Combinatorial proof: The LHS is the number of 2-element subsets of a 2-\(n\)-element set. To see that this is the same as the RHS, we notice that if we split the 2-\(n\)-element set \(A\) into 2 equally sized sets \(B\) and \(C\), any 2-element subset of \(A\) is either a 2-element subset of \(B\), a 2-element subset of \(C\), or has one element from each of \(B\) and \(C\). There are \(\binom{n}{2}\) 2-element subsets of \(B\) and \(C\), and \(n^2\) ways to choose one element from each of \(B\) and \(C\), so this means there are \(2 \binom{n}{2} + n^2\) 2-element subsets of \(A\).

Algebraic proof:

\[
\binom{2n}{2} = \frac{2n!}{2!(2n-2)!} = \frac{2n(2n-1)}{2} = \frac{4n^2 - 2n}{2} = 2n^2 - n = n^2 - n + n^2 = 2 \frac{n(n-1)}{2} + n^2 = 2 \frac{n}{2} + n^2.
\]

5. Give a combinatorial proof that

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.
\]

(Hint: rearrange the equation so you don’t have any negative numbers.)
Solution. Combinatorial proofs are much more natural when you have nonnegative numbers, so we rearrange the equation as

\[
\sum_{0 \leq k \leq n} \binom{n}{k} = \sum_{0 \leq k \leq n} \binom{n}{k}.
\]

Let \(A\) be a set of size \(n\). The LHS is the number of subsets of \(A\) whose size is even; the RHS is the number of subsets of \(A\) whose size is odd. So to show this equality, we should give a bijection between even-sized subsets of \(A\) and odd-sized subsets of \(A\).

If \(n\) is odd, then let \(f : \{B \subseteq A \mid |B| \text{ is even}\} \to \{B \subseteq A \mid |B| \text{ is odd}\}\) be the function such that \(f(B) = \overline{B}\). \(f\) is a bijection because the function \(g : \{B \subseteq A \mid |B| \text{ is odd}\} \to \{B \subseteq A \mid |B| \text{ is even}\}\) where \(g(C) = \overline{C}\) is its inverse.

If \(n\) is even, then we pick some element \(a \in A\). Since \(A' = A - \{a\}\) has odd size, by the above argument we have a one-to-one correspondence between even- and odd-sized subsets of \(A\) which sends each subset to its complement. This means we just have to establish a one-to-one correspondence between the remaining even-sized subsets of \(A\) (that is, the even-sized subsets of \(A\) that contain \(a\)) and the remaining odd-sized subsets of \(A\) (that is, the odd-sized subsets of \(A\) that contain \(a\)). So we define \(f : f : \{B \subseteq A \mid |B| \text{ is even}\} \to \{B \subseteq A \mid |B| \text{ is odd}\}\) as follows. If \(X \subseteq A'\), then \(f(X) = A' - X\). If \(X = X' \cup \{a\}\), where \(X' \subseteq A'\), then \(f(X) = \overline{X'} \cup \{a\}\). We leave it to the reader to check that \(f\) is well-defined (i.e. \(f(X)\) has odd size for all \(X\)) and is a bijection.