Concepts

- product & sum rule for counting
- principle of inclusion-exclusion
- pigeonhole principle

Problem Solutions

1. Let $X$ be the set of “words” in $\{a, b, c, d\}$ of length $n$.

   (a) How many elements of $X$ are palindromes?

   Solution. The answer depends on the parity of $n$. Say $n = 2k$ for some $k \in \mathbb{Z}$. Then a palindromic word of length $n$ is completely determined by the first $k$ letters. There are 4 choices for the first letter, and for every choice of first letter, 4 choices for the second, etc. So by the product rule for counting, there are $4^{n/2}$ palindromic words of length $n$.

   If $n = 2k + 1$, then a palindromic word is completely determined by the first $k + 1$ letters. Since there are 4 choices for each letter, and each choice is independent of the others, there are $4^{(n+1)/2}$ palindromic words of length $n$.

   (b) How many elements of $X$ contain exactly one $a$?

   Solution. To make a word containing exactly one $a$, we first choose the position of the $a$ and then create the rest of the word. There are $n$ possible positions for $a$. Regardless of where we put the $a$, there are $n - 1$ other positions that need to be filled with $b, c, d$. Since there are 3 choices for how to fill each of these positions, there are $3^n$ words with exactly one $a$.

2. How many functions are there from a finite set $A$ to the set $\{0, 1\}$? Does this number look familiar? How can you relate functions $f : A \to \{0, 1\}$ to subsets of $A$?

   Solution. There are $2^{|A|}$ functions from $A$ to $\{0, 1\}$: there are 2 choices for where to send the first element, 2 choices for where to send the second, etc.

   This is the size of the power set of $A$. Given a subset $B$, you can create a function $f : A \to \{0, 1\}$ by saying $f(a) = 1$ if $a \in B$ and $f(a) = 0$ otherwise. This creates a unique function for every subset. Given a function $f : A \to \{0, 1\}$, one way you can get a subset is by considering $f^{-1}(1)$. This gives a unique subset for every function.

3. Let $A$ be the set of integers divisible by 4 and 6. What is $|A \cap [0, 100]|$?

   Solution. Let $B$ be the set of integers in $[0, 100]$ divisible by 4 and $C$ be the set of integers in $[0, 100]$ divisible by 6. $A = B \cup C$, and we would like to compute its size.

   Since $B$ and $C$ are not disjoint, $|B \cup C| = |B| + |C| - |B \cap C|$ (this is because if we just add the cardinalities of $B$ and $C$, we count the elements in their intersection twice). $|B| = 26$, since $4m \in [0, 100]$ for $0 \leq m \leq 25$. $|C| = 17$, since $6m \in [0, 100]$ for $0 \leq m \leq 16$. $B \cap C$ consists of the integers in $[0, 100]$ that are common multiples of 4 and 6. The lcm of 4 and 6 is 12, so the elements of $B \cap C$ are the multiples of 12 in $[0, 100]$. There are 9 of these, since $12m \in [0, 100]$ for $0 \leq m \leq 8$.

   So $|A| = 26 + 17 - 9 = 34$. 
4. Suppose you have 6 people at a party. Show that there are at least 2 people who know the same number of people at the party. (Hint: can you have both a person who knows no one and a person who knows everyone at the party?)

Solution. Let $x_i$ be the number of people that person $i$ knows. By the hint, either $x_i \in \{0, 1, 2, 3, 4\}$ or $x_i \in \{1, 2, 3, 4, 5\}$. In each case, there are 6 people and 5 possible values for $x_i$, so by the pigeonhole principle, there exist $i$ and $j$ such that $x_i = x_j$.

5. Suppose $A, B, C$ are finite sets whose cardinalities you know. How can you write $|A \cup B \cup C|$ in terms of the cardinalities of $A, B, C$ and their intersections? (Hint: try it with 2 sets first.)

Solution. As mentioned in the solution of problem 3, for two sets $X$ and $Y$, $|X \cup Y| = |X| + |Y| - |X \cap Y|$. Let’s call this equality $(\ast)$. To find the size of $A \cup B \cup C$, we apply this result a number of times, first thinking of $A \cup B$ as one set and $C$ as another.

So we have

$$|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$$

now applying $(\ast)$ to $A \cup B$ and simplifying

$$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$$

now applying $(\ast)$ to $(A \cap C) \cup (B \cap C)$

$$= |A| + |B| - |A \cap B| + |C| - ((|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|))$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap B \cap C)|$$

6. Let $d \in \mathbb{Z}^+$. What is the smallest number of integers you need to guarantee that at least 3 of them have the same remainder when divided by $d$?

Solution. There are $d$ possible remainders when you divide by $d$, so this problem can be translated to: we have $d$ pigeonholes, so how many pigeons we need to guarantee that a hole has at least 3 pigeons in it?

The answer is $2d + 1$. If you only have $2d$, it’s possible for no hole to have 3 pigeons (for example, if you put 2 pigeons in each hole). But by the generalized pigeonhole principle, if you have $2d + 1$ pigeons, then some hole has at least $\lceil (2d + 1)/d \rceil = 3$ pigeons.