Problem Solutions

1. Prove or disprove the following:

(a) $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

Solution. This equivalence is true; we show that if the LHS is true, then the RHS is true, and vice versa.

Suppose $\exists x(P(x) \vee Q(x))$ is true. Let c be an element of the domain such that $P(c) \vee Q(c)$ is true. Then by the definition of disjunction, either $P(c)$ is true or $Q(c)$ is true. In the first case, we have that $\exists x P(x)$ is true by existential generalization; in the second, we have that $\exists x Q(x)$ is true. Thus $\exists x P(x) \vee \exists x Q(x)$ is true.

Suppose $\exists x P(x) \vee \exists x Q(x)$ is true. Then either there is a c in the domain such that $P(c)$ is true or there is a d in the domain such that $Q(d)$ is true. In the first case $P(c) \vee Q(c)$ is true; in the second, $P(d) \vee Q(d)$ is true. So $\exists x(P(x) \vee Q(x))$ is true.

(b) $\exists x(P(x) \wedge Q(x)) \equiv \exists x P(x) \wedge \exists x Q(x)$

Solution. This equivalence is false; the LHS implies the RHS, but the RHS does not imply the LHS. Let $P(x)$ be the statement "x is even" and Q(x) be the statement "x is odd", where the domain is $\mathbb{Z}$. Then the RHS is true, but the LHS is false. This shows that the two statements are not logically equivalent, since if they were they would have to have the same truth value for any choice of $P(x), Q(x)$ and domain.

2. Let $f : A \to B$ be a function and let $S \subseteq B$. Show that $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$.

Solution. First, note that $f^{-1}(\overline{S}) = \{x \in A | f(x) \in \overline{S}\} = \{x \in A | f(x) \notin S\}$.

Suppose $x \in f^{-1}(\overline{S})$. Then $f(x) \notin S$. Since $f^{-1}(S) = \{x \in A | f(x) \in S\}$, this implies $x \notin f^{-1}(S)$. By the definition of complement, this means $x \in \overline{f^{-1}(S)}$. This shows $f^{-1}(\overline{S}) \subseteq \overline{f^{-1}(S)}$.

To see the opposite inclusion, let $x \in \overline{f^{-1}(S)}$. Then $x \notin f^{-1}(S)$, which implies that $f(x) \notin S$. In other words, $f(x) \in \overline{S}$, which means $x \in f^{-1}(\overline{S})$.

3. Let d be a fixed positive integer. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function that takes a to a **div** d and $g : \mathbb{Z} \to \mathbb{Z}$ be the function that takes a to a **mod** d. Are either of these functions one-to-one? Onto?

Solution. Neither function is one-to-one. $f(2d) = f(2d+1) = 2$, and $g(d) = g(2d) = 0$. f is onto, since for $k \in \mathbb{Z}$, $f(kd) = k$. However, g is not, since by definition, $0 \leq a \textbf{ mod } d < d$ for all $a \in \mathbb{Z}$.

4. Show that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Solution. Draw the elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ in the plane and use the same path as you use to show $\mathbb{Q}$ is countable.

5. Prove that there are infinitely many primes.

Solution. (From Lecture 7) Assume, for the sake of contradiction, that there are only finitely many primes $p_1, \ldots, p_n$.

Consider the positive integer $q = p_1 p_2 \cdots p_n + 1$. By the Fundamental Theorem of Arithmetic, q is either prime or can be written as a product of 2 or more primes.

If $p_i | q$, then since p_i also divides $p_1 p_2 \cdots p_n$, p_i divides $q - p_1 p_2 \cdots p_n = 1$. This is a contradiction, so no prime divides q. This means that q is prime. On the other hand, $q \neq p_i$ for any i, because $q > p_i$ for all i. So q is a prime not on the list above of all primes, a contradiction. Thus, our assumption that there are only finitely many primes is false.