Problem Solutions

1. Let A and B be sets, and $f : A \to B$ a function between them. For $S, T \subseteq A$, show that $f(S \cap T) \subseteq f(S) \cap f(T)$. Find an example for which this inclusion is proper.

Solution. $f(S \cap T) = \{f(x) | x \in S \cap T\}$. If $x \in S \cap T$, then $x \in S$ and $x \in T$, so $f(x) \in f(S)$ and $f(x) \in f(T)$. This implies $f(x) \in f(S) \cap f(T)$, which completes the proof.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$, $S = [-1, 0]$, and $T = [0, 1]$. Then $S \cap T = \{0\}$, $f(S \cap T) = 0$, and $f(S) \cap f(T) = [0, 1]$.

2. Give an example of a function $f : \mathbb{Q} \to \mathbb{Q}$ that is:

(a) bijective

Solution. Let f be the function that sends 0 to 0 and sends r to $1/r$ for $r \neq 0$. f is a bijection: $1/r = 1/s$ if and only if $r = s$, so f is injective, and for $r \in \mathbb{Q} - \{0\}$, $f(1/r) = r$ (and $f(0) = 0$), so f is surjective.

(b) one-to-one but not onto

Solution. Let f be the function $f(x) = x^3$. It is not onto because 2 is not in the range. It is injective because it is a strictly increasing function: if $x > y$, then $f(x) > f(y)$. So the only way for $f(x) = f(y)$ is for $x = y$.

(c) surjective but not injective

Solution. Let f be the function that acts as the identity on non-integers and non-positive integers, and sends each positive integer n to $n - 1$. Then it is not injective, because $f(0) = f(1) = 0$. Non-integers and non-positive integers are clearly in the range; each positive integer n is also in the range, since $f(n + 1) = n$. Thus, it is surjective.

(d) neither one-to-one nor onto.

Solution. Let f be the function $f(x) = x^2$. It is not onto, since the range consists of non-negative rationals. It is not one-to-one since $f(1) = f(-1) = 1$.

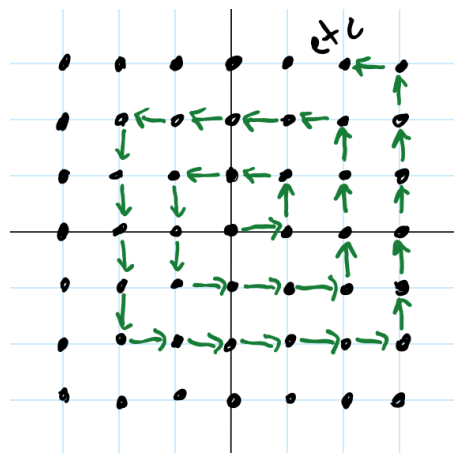
3. Find a simple formula or rule to describe the nth term of the following sequences.

(a) $-1, 5, -9, 13, -17, \ldots$: $a_n = (-1)^n(4n - 3)$

(b) $8, 10, 13, 17, 22, 28, \ldots$: $a_n = n + a_{n-1}$

(c) $2, 16, 54, 128, 250, 432, 686 \ldots$: $a_n = 2n^3$

4. Show that if $f : A \to B$ and $g : B \to C$ are bijections, then so is $g \circ f$.



Solution. First, we show $g \circ f$ is one-to-one. Suppose $x, y \in A$ and $g(f(x)) = g(f(y))$. Since g is a bijection, and in particular is injective, this implies $f(x) = f(y)$. As f is a bijection, this implies $x = y$.

Next, we show surjectivity. Let $z \in C$. We would like to produce an element of A that is sent to z by $g \circ f$. Since g is surjective, there exists $y \in B$ such that $g(y) = z$. Since f is surjective and y is in the codomain of f, there exists $x \in A$ such that $f(x) = y$. Now we have $g(f(x)) = g(y) = z$.

5. Let $2\mathbb{Z}$ denote the set of even integers. Show that $\mathbb{Z} \times \mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality. (Hint: you may want to compose a number of different bijections)

Solution. First, we find a bijection between $2\mathbb{Z}$ and $\mathbb{Z}$, and another bijection between $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}^+$. From lecture, there is a bijection from $\mathbb{Z}^+$ to $\mathbb{Z}$, so by composing these bijections, we'll get a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $2\mathbb{Z}$.

Let $f : \mathbb{Z} \to 2\mathbb{Z}$ be the function $f(x) = 2x$. f is injective since if $f(x) = f(y)$, then $2x = 2y$, which implies $x = y$. f is also surjective, since for $y \in 2\mathbb{Z}$, $y = 2x$ for some $x \in 2\mathbb{Z}$ by the definition of odd. So $f(x) = 2x = y$.

To describe $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$, we draw a picture (above). We draw an infinite path (in green) starting at $(0, 0)$ and winding about the origin. g sends the nth point in this path to n. Intuitively speaking, it's clear that this function is a bijection; to really prove it, we'd need to write down a more concrete rule for where g sends the point (i, j).

Let $h : \mathbb{Z}^+ \to \mathbb{Z}$ be the bijection described in lecture.

Then $f \circ h \circ g : \mathbb{Z} \times \mathbb{Z} \to 2\mathbb{Z}$ is a bijection, since a composition of bijections is a bijection, which implies the two sets have the same cardinality.

6. Suppose A and B are disjoint countable sets. Is $A \cup B$ countable? $A \times B$?

Solution. The answer to both questions is yes.

Let $f : A \to \mathbb{Z}^+$ and $g : B \to \mathbb{Z}^+$ be bijections (these functions exist by the assumption that A and B are countable). Let $h : A \cup B \to \mathbb{Z}$ be the function that takes an element $a \in A$ to $-f(a)$ and takes an element $b \in B$ to $g(b) - 1$. h is well-defined, since A and B are disjoint.

Injectivity of h: if $h(x) = h(y)$ for some $x, y \in A \cup B$, then either both x and y are in A or they're both in B, since $h(A)$ consists of negative numbers and $h(B)$ consists of nonnegative numbers. It then follows from the injectivity of f and g that $x = y$.

The surjectivity of h follows immediately from the fact that f and g have inverses. For example, let $n \in \mathbb{Z}$ be nonnegative. We want $x \in A \cup B$ such that $h(x) = n$; since $n \geq 0$, x would be in B, so in fact we are looking for $x \in B$ such that $g(x) - 1 = n$ or, equivalently, $g(x) = n + 1$. Applying g^{-1} to both sides gives that $x = g^{-1}(n+1)$. The argument for surjectivity of h on negative integers is similar.

Let $p : A \to \mathbb{Z}$ and $q : B \to \mathbb{Z}$. Note that these functions exist because $\mathbb{Z}$ is countable. Let $r : A \times B \to \mathbb{Z} \times \mathbb{Z}$ be the function that takes $(a, b) \in A \times B$ to $(p(a), q(b)) \in \mathbb{Z} \times \mathbb{Z}$. From Problem 5, $\mathbb{Z} \times \mathbb{Z}$ is countable, so to show $A \times B$ is countable, it suffices to show that r is a bijection.

Injectivity of r: Suppose $r(a, b) = r(c, d)$ for $(a, b), (c, d) \in A \times B$. Then $(p(a), q(b)) = (p(c), q(d))$, which is true if and only if $p(a) = p(c)$ and $q(b) = q(d)$. Since p and q are injective, this implies $a = c$ and $b = d$, which in turn implies $(a, b) = (c, d)$.

Surjectivity of r: Let $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Then by the surjectivity of p and q, there exists $x \in A$ and $y \in B$ such that $p(x) = m$ and $q(y) = n$. $r(x, y) = (m, n)$, so r is surjective.