Problem Solutions

1. Let $A$ and $B$ be sets, and $f : A \to B$ a function between them. For $S, T \subseteq A$, show that $f(S \cap T) \subseteq f(S) \cap f(T)$. Find an example for which this inclusion is proper.

Solution. $f(S \cap T) = \{ f(x) | x \in S \cap T \}$. If $x \in S \cap T$, then $x \in S$ and $x \in T$, so $f(x) \in f(S)$ and $f(x) \in f(T)$. This implies $f(x) \in f(S) \cap f(T)$, which completes the proof.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$, $S = [-1, 0]$, and $T = [0, 1]$. Then $S \cap T = \{0\}$, $f(S \cap T) = 0$, and $f(S) \cap f(T) = [0, 1]$.

2. Give an example of a function $f : \mathbb{Q} \to \mathbb{Q}$ that is:

(a) bijective

Solution. Let $f$ be the function that sends 0 to 0 and sends $r$ to $1/r$ for $r \neq 0$. $f$ is a bijection: $1/r = 1/s$ if and only if $r = s$, so $f$ is injective, and for $r \in \mathbb{Q} - \{0\}$, $f(1/r) = r$ (and $f(0) = 0$), so $f$ is surjective.

(b) one-to-one but not onto

Solution. Let $f$ be the function $f(x) = x^3$. It is not onto because 2 is not in the range. It is injective because it is a strictly increasing function: if $x > y$, then $f(x) > f(y)$. So the only way for $f(x) = f(y)$ is for $x = y$.

(c) surjective but not injective

Solution. Let $f$ be the function that acts as the identity on non-integers and non-positive integers, and sends each positive integer $n$ to $n - 1$. Then it is not injective, because $f(0) = f(1) = 0$. Non-integers and non-positive integers are clearly in the range; each positive integer $n$ is also in the range, since $f(n + 1) = n$. Thus, it is surjective.

(d) neither one-to-one nor onto.

Solution. Let $f$ be the function $f(x) = x^2$. It is not onto, since the range consists of non-negative rationals. It is not one-to-one since $f(1) = f(-1) = 1$.

3. Find a simple formula or rule to describe the $n^{th}$ term of the following sequences.

(a) $-1, 5, -9, 13, -17, \ldots$: $a_n = (-1)^n(4n - 3)$

(b) $8, 10, 13, 17, 22, 28, \ldots$: $a_n = n + a_{n-1}$

(c) $2, 16, 54, 128, 250, 432, 686 \ldots$: $a_n = 2n^3$

4. Show that if $f : A \to B$ and $g : B \to C$ are bijections, then so is $g \circ f$. 

Solution. First, we show \( g \circ f \) is one-to-one. Suppose \( x, y \in A \) and \( g(f(x)) = g(f(y)) \). Since \( g \) is a bijection, and in particular is injective, this implies \( f(x) = f(y) \). As \( f \) is a bijection, this implies \( x = y \).

Next, we show surjectivity. Let \( z \in C \). We would like to produce an element of \( A \) that is sent to \( z \) by \( g \circ f \). Since \( g \) is surjective, there exists \( y \in B \) such that \( g(y) = z \). Since \( f \) is surjective and \( y \) is in the codomain of \( f \), there exists \( x \in A \) such that \( f(x) = y \). Now we have \( g(f(x)) = g(y) = z \).

5. Let \( 2\mathbb{Z} \) denote the set of even integers. Show that \( \mathbb{Z} \times \mathbb{Z} \) and \( 2\mathbb{Z} \) have the same cardinality.

(Hint: you may want to compose a number of different bijections)

Solution. First, we find a bijection between \( 2\mathbb{Z} \) and \( \mathbb{Z} \), and another bijection between \( \mathbb{Z} \times \mathbb{Z} \) and \( \mathbb{Z}^+ \). From lecture, there is a bijection from \( \mathbb{Z}^+ \) to \( \mathbb{Z} \), so by composing these bijections, we’ll get a bijection from \( \mathbb{Z} \times \mathbb{Z} \) to \( 2\mathbb{Z} \).

Let \( f : \mathbb{Z} \rightarrow 2\mathbb{Z} \) be the function \( f(x) = 2x \). \( f \) is injective since if \( f(x) = f(y) \), then \( 2x = 2y \), which implies \( x = y \). \( f \) is also surjective, since for \( y \in 2\mathbb{Z} \), \( y = 2x \) for some \( x \in \mathbb{Z} \) by the definition of odd. So \( f(x) = 2x = y \).

To describe \( g : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}^+ \), we draw a picture (above). We draw an infinite path (in green) starting at \((0,0)\) and winding about the origin. \( g \) sends the \( n \)th point in this path to \( n \). Intuitively speaking, it’s clear that this function is a bijection; to really prove it, we’d need to write down a more concrete rule for where \( g \) sends the point \((i,j)\).

Let \( h : \mathbb{Z}^+ \rightarrow \mathbb{Z} \) be the bijection described in lecture.

Then \( f \circ h \circ g : \mathbb{Z} \times \mathbb{Z} \rightarrow 2\mathbb{Z} \) is a bijection, since a composition of bijections is a bijection, which implies the two sets have the same cardinality.

6. Suppose \( A \) and \( B \) are disjoint countable sets. Is \( A \cup B \) countable? \( A \times B \)?

Solution. The answer to both questions is yes.

Let \( f : A \rightarrow \mathbb{Z}^+ \) and \( g : B \rightarrow \mathbb{Z}^+ \) be bijections (these functions exist by the assumption that \( A \) and \( B \) are countable). Let \( h : A \cup B \rightarrow \mathbb{Z} \) be the function that takes an element \( a \in A \) to \( -f(a) \) and takes an element \( b \in B \) to \( g(b) - 1 \). \( h \) is well-defined, since \( A \) and \( B \) are disjoint.
Injectivity of $h$: if $h(x) = h(y)$ for some $x, y \in A \cup B$, then either both $x$ and $y$ are in $A$ or they’re both in $B$, since $h(A)$ consists of negative numbers and $h(B)$ consists of nonnegative numbers. It then follows from the injectivity of $f$ and $g$ that $x = y$.

The surjectivity of $h$ follows immediately from the fact that $f$ and $g$ have inverses. For example, let $n \in \mathbb{Z}$ be nonnegative. We want $x \in A \cup B$ such that $h(x) = n$; since $n \geq 0$, $x$ would be in $B$, so in fact we are looking for $x \in B$ such that $g(x) - 1 = n$ or, equivalently, $g(x) = n + 1$. Applying $g^{-1}$ to both sides gives that $x = g^{-1}(n+1)$. The argument for surjectivity of $h$ on negative integers is similar.

Let $p : A \rightarrow \mathbb{Z}$ and $q : B \rightarrow \mathbb{Z}$. Note that these functions exist because $\mathbb{Z}$ is countable. Let $r : A \times B \rightarrow \mathbb{Z} \times \mathbb{Z}$ be the function that takes $(a, b) \in A \times B$ to $(p(a), q(b)) \in \mathbb{Z} \times \mathbb{Z}$. From Problem 5, $\mathbb{Z} \times \mathbb{Z}$ is countable, so to show $A \times B$ is countable, it suffices to show that $r$ is a bijection.

Injectivity of $r$: Suppose $r(a, b) = r(c, d)$ for $(a, b), (c, d) \in A \times B$. Then $(p(a), q(b)) = (p(c), q(d))$, which is true if and only if $p(a) = p(c)$ and $q(b) = q(d)$. Since $p$ and $q$ are injective, this implies $a = c$ and $b = d$, which in turn implies $(a, b) = (c, d)$.

Surjectivity of $r$: Let $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Then by the surjectivity of $p$ and $q$, there exists $x \in A$ and $y \in B$ such that $p(x) = m$ and $q(y) = n$. $r(x, y) = (m, n)$, so $r$ is surjective.