Problem Solutions

1. Find the power sets of the following sets.

(a) $\{\{1,2\}\}$

(b) $\{1,\{2,3\}\}$

(c) $\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$

Solution. (a) $\mathcal{P}(\{\{1,2\}\}) = \{\emptyset,\{\{1,2\}\}\}$

(b) $\mathcal{P}(\{1,\{2,3\}\}) = \{\emptyset,\{1\},\{\{2,3\}\},\{1,\{2,3\}\}\}$

(c) $\mathcal{P}(\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}) = \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\{\emptyset,\{\emptyset\}\}\},\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset,\{\emptyset\}\}\},\{\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}\}$

2. Let A be the set of Cal student who like rain, B be the set of Cal students who have never seen snow, and C the set of Cal students who wish they lived by the beach. Describe the students in each of the following combinations of sets and draw the corresponding Venn diagram.

(a) $A \cap (B - C)$

(b) $\overline{A} \cup \overline{B} \cup \overline{C}$

(c) $B - (A \cap \overline{C})$

Solution. (a) $A \cap (B - C)$: This is the set of students who like rain, have never seen snow, and don't wish they lived by the beach.

(b) $\overline{A} \cup \overline{B} \cup \overline{C}$: This is the set of students who either don't like the rain, have seen snow, or don't wish they lived by the beach.

(c) $B - (A \cap \overline{C})$: This is the set of students who have never seen snow and either don't like the rain or wish they lived by the beach.

3. Are $A \times B \times C$ and $(A \times B) \times C$ equal? Why or why not? What about $A \cup (B \cup C)$ and $(A \cup B) \cup C$?

Solution. The first pair of sets are not equal; in fact, they are disjoint. Elements of $A \times B \times C$ are ordered triples (a, b, c) with $a \in A, b \in B$, and $c \in C$. Elements of $(A \times B) \times C$ are ordered pairs (d, c) where $d \in A \times B$ and $c \in C$.

The second pair of sets are equal. $A \cup (B \cup C) = \{x : x \in A \ \vee (x \in B \vee x \in C)\}$. By the associative property of "or", this is the same as $\{x : (x \in A \vee x \in B) \ \vee x \in C)\} = (A \cup B) \cup C$.

4. (De Morgan's laws) Show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by showing that each set contains the other.

Solution. First, we show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Let $x \in \overline{A \cap B}$. Then by the definition of complement, $x \notin A \cap B$. This implies that $x \notin A$ or $x \notin B$, or, in other words, $x \in \overline{A}$ or $x \in \overline{B}$. By the definition of union, $x \in \overline{A} \cup \overline{B}$.

Second, we show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Let $x \in \overline{A} \cup \overline{B}$. Then either $x \in \overline{A}$ or $x \in \overline{B}$, by the definition of union. In other words, $x \notin A$ or $x \notin B$, which implies $x \notin A \cap B$. By the definition of complement, this implies $x \in \overline{A \cap B}$.

5. Determine if these functions are injective or surjective. Justify your answers

(a) $f : \mathbb{R} \to \mathbb{Q}$ where $f(x) = \lceil x \rceil$

(b) $g : \mathbb{Z} \to \mathbb{Q}$ where $g(x) = \lceil x \rceil$

(c) $h : \mathbb{R} \to \mathbb{Z}$ where $h(x) = \lceil x \rceil$

Are f and h the same function?

Solution. (a) Neither injective ($f(1.1) = f(1.5)$) nor surjective ($\neg\exists x \in \mathbb{Z}(g(x) = 1/2)$).

(b) Injective ($g(x) = x$ for all $x \in \mathbb{Z}$, so if $g(x) = g(y)$, then $x = y$), but not surjective ($\neg\exists x \in \mathbb{Z}(g(x) = 1/2)$)

(c) Not injective ($h(1.1) = h(1.5)$) but surjective ($\forall x \in \mathbb{Z}(h(x) = x)$, so the range of h is $\mathbb{Z}$). They are not the same function, because they do not have the same codomain.

6. Let f be the function that assigns to each bit string the number of 1's in the string minus the number of 0's. Find the domain (i.e. the set of elements assigned a value by f) of f and a possible codomain. What choice of codomain makes f onto? Is f one-to-one?

Solution. The domain is the set of all bit strings. A possible codomain is $\mathbb{R}$. The range of f is $\mathbb{Z}$ (f sends the string of n 1's to n, the string of n 0's to $-n$ and the empty string to 0), so that choice of codomain will make f onto. f is not one-to-one, since the string 10 and 01 are both sent to 0.