Problem Solutions

1. Find the power sets of the following sets.
   (a) \(\{1, 2\}\)
   (b) \(\{1, \{2, 3\}\}\)
   (c) \(\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\)

   **Solution.**
   (a) \(P(\{\{1, 2\}\}) = \emptyset, \{\{1, 2\}\}\)
   (b) \(P(\{1, \{2, 3\}\}) = \emptyset, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\)
   (c) \(P(\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}) = \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\)

2. Let \(A\) be the set of Cal student who like rain, \(B\) be the set of Cal students who have never seen snow, and \(C\) the set of Cal students who wish they lived by the beach. Describe the students in each of the following combinations of sets and draw the corresponding Venn diagram.
   (a) \(A \cap (B - C)\)
   (b) \(A \cup B \cup C\)
   (c) \(B - (A \cap C)\)

   **Solution.**
   (a) \(A \cap (B - C)\): This is the set of students who like rain, have never seen snow, and don’t wish they lived by the beach.
   (b) \(A \cup B \cup C\): This is the set of students who either don’t like the rain, have seen snow, or don’t wish they lived by the beach.
   (c) \(B - (A \cap C)\): This is the set of students who have never seen snow and either don’t like the rain or wish they lived by the beach.

3. Are \(A \times B \times C\) and \((A \times B) \times C\) equal? Why or why not? What about \(A \cup (B \cup C)\) and \((A \cup B) \cup C\)?

   **Solution.**
   The first pair of sets are not equal; in fact, they are disjoint. Elements of \(A \times B \times C\) are ordered triples \((a, b, c)\) with \(a \in A, b \in B,\) and \(c \in C\). Elements of \((A \times B) \times C\) are ordered pairs \((d, c)\) where \(d \in A \times B\) and \(c \in C\).

   The second pair of sets are equal. \(A \cup (B \cup C) = \{x : x \in A \lor (x \in B \lor x \in C)\}\). By the associative property of “or”, this is the same as \(\{x : (x \in A \lor x \in B) \lor x \in C)\} = (A \cup B) \cup C\).

4. (De Morgan’s laws) Show that \(\overline{A \cap B} = \overline{A} \cup \overline{B}\) by showing that each set contains the other.

   **Solution.**
   First, we show \(\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}\). Let \(x \in \overline{A \cap B}\). Then by the definition of complement, \(x \notin A \cap B\). This implies that \(x \notin A\) or \(x \notin B\), or, in other words, \(x \in \overline{A}\) or \(x \in \overline{B}\). By the definition of union, \(x \in \overline{A} \cup \overline{B}\).

   Second, we show \(\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}\). Let \(x \in \overline{A} \cup \overline{B}\). Then either \(x \in \overline{A}\) or \(x \in \overline{B}\), by the definition of union. In other words, \(x \notin A\) or \(x \notin B\), which implies \(x \notin A \cap B\). By the definition of complement, this implies \(x \in \overline{A \cap B}\).
5. Determine if these functions are injective or surjective. Justify your answers.

(a) \( f : \mathbb{R} \to \mathbb{Q} \) where \( f(x) = \lceil x \rceil \)

(b) \( g : \mathbb{Z} \to \mathbb{Q} \) where \( g(x) = \lceil x \rceil \)

(c) \( h : \mathbb{R} \to \mathbb{Z} \) where \( h(x) = \lceil x \rceil \)

Are \( f \) and \( h \) the same function?

Solution. (a) Neither injective \((f(1.1) = f(1.5))\) nor surjective \((\neg \exists x \in \mathbb{Z}(g(x) = 1/2))\).

(b) Injective \((g(x) = x \text{ for all } x \in \mathbb{Z}, \text{ so if } g(x) = g(y), \text{ then } x = y)\), but not surjective \((\neg \exists x \in \mathbb{Z}(g(x) = 1/2))\)

(c) Not injective \((h(1.1) = h(1.5))\) but surjective \((\forall x \in \mathbb{Z}(h(x) = x), \text{ so the range of } h \text{ is } \mathbb{Z})\).

They are not the same function, because they do not have the same codomain.

6. Let \( f \) be the function that assigns to each bit string the number of 1’s in the string minus the number of 0’s. Find the domain (i.e. the set of elements assigned a value by \( f \)) of \( f \) and a possible codomain. What choice of codomain makes \( f \) onto? Is \( f \) one-to-one?

Solution. The domain is the set of all bit strings. A possible codomain is \( \mathbb{R} \). The range of \( f \) is \( \mathbb{Z} \) (\( f \) sends the string of \( n \) 1’s to \( n \), the string of \( n \) 0’s to \( -n \) and the empty string to 0), so that choice of codomain will make \( f \) onto. \( f \) is not one-to-one, since the string 10 and 01 are both sent to 0.