Problem Solutions

1. Let $P(x)$ be the statement "x lives north of Dwight," where the domain of x consists of all students in your small group. Find the truth values of $P(x)$ for all x in the domain.

2. Choose 3 propositional functions $F(j)$, $G(j)$, and $H(j)$ where the domain of j is all dogs. Translate the following statements into English and find their truth value.

(a) $\forall j F(j)$

(b) $\exists j(G(j) \to H(j))$

(c) $\forall j((F(j) \vee H(j)) \to \neg G(j))$

(d) $\exists j(\neg F(j) \wedge G(j)) \vee \forall j H(j))$

Solution. Let $F(j)$ be "j is a good dog," $G(j)$ be "j barks wildly at street-cleaning trucks," and $H(j)$ be "j does not like walks."

(a) Every dog is a good dog. True.

(b) There is a dog that either does not like walks or does not bark wildly at street-cleaning trucks. True (probably). (This is using the logical equivalence $p \to q \equiv \neg p \vee q$.)

(c) Dogs either do not bark wildly at street-cleaning trucks or are bad dogs who like walks. False. (Using De Morgan's laws and $p \to q \equiv \neg p \vee q$).

(d) Either there is a bad dog who barks wildly at street-cleaning trucks or no dogs like walks. False because all dogs are good dogs and at least one dog likes walks.

3. Let $S(x, y)$ be the statement "x has asked y a question" and let $P(x)$ be the statement "x is a student," where the domain is all Math 55 GSIs and students and Professor Williams. Translate the following statements into symbols.

(a) Professor Williams has asked Melissa a question.

(b) Professor Williams has asked everyone a question.

(c) At least one student has asked another student a question.

(d) Nobody has asked every student a question.

(e) There is a student who has never been asked a question.

Solution. (a) $S(\text{Professor Williams, Melissa})$

(b) $\forall y S(\text{Professor Williams}, y)$

(c) $\exists x \exists y(P(x) \wedge P(y) \wedge S(x, y))$

(d) $\neg \exists x(\forall y(P(y) \wedge S(x, y)))$

(e) $\exists y(P(y) \wedge \forall x \neg S(x, y))$ or, equivalently, $\exists y(P(y) \wedge \neg \exists x S(x, y))$

4. Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.

Solution. To show that these statements are not logically equivalent, we need to find a domain for the two statements such that they have different truth values on that domain.

Let the domain consist of two elements, j and k such that $P(j)$ is true, $P(k)$ is false, $Q(j)$ is false, and $Q(k)$ is true. Then the statement $\forall x(P(x) \vee Q(x))$ is true. However, j is a counterexample to $\forall x Q(x)$ and k is a counterexample to $\forall x P(x)$ so the disjunction $\forall x P(x) \vee \forall x Q(x)$ is false.

5. Express the negations of each of these statements so that all negation symbols are in front of predicates. (Hint: remember De Morgan's Laws)

(a) $\neg(\forall x \exists y P(x, y) \ \wedge \ \forall x \exists y Q(x, y))$

Solution.

$$\begin{aligned}
\neg(\forall x \exists y P(x, y) \ \wedge \ \forall x \exists y Q(x, y)) &\equiv \neg(\forall x \exists y P(x, y)) \ \vee \ \neg(\forall x \exists y Q(x, y)) \\
&\equiv \exists x(\neg \exists y P(x, y)) \ \vee \ \exists x(\neg \exists y Q(x, y)) \\
&\equiv \exists x \forall y \neg P(x, y) \ \vee \ \exists x \forall y \neg Q(x, y)
\end{aligned}$$

(b) $\neg \exists y(S(y) \ \vee \ \forall x \neg R(x, y))$

Solution.

$$\begin{aligned}
\neg \exists y(S(y) \ \vee \ \forall x \neg R(x, y)) &\equiv \forall y \neg(S(y) \ \vee \ \forall x \neg R(x, y)) \\
&\equiv \forall y(\neg S(y) \wedge \neg \forall x \neg R(x, y)) \\
&\equiv \forall y(\neg S(y) \wedge \exists x R(x, y))
\end{aligned}$$

6. Consider the argument "If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$." Is it valid? What if the last two sentences were instead "Suppose that $n^2 < 1$. Then $n < 1$."?

Solution. The first version is not a valid argument; it is the fallacy that $((p \to q) \wedge q) \to p$, which is not a tautology. The second version is a valid argument; it is *modus tollens.*