Problem Solutions

1. Let $P(x)$ be the statement “$x$ lives north of Dwight,” where the domain of $x$ consists of all students in your small group. Find the truth values of $P(x)$ for all $x$ in the domain.

2. Choose 3 propositional functions $F(j)$, $G(j)$, and $H(j)$ where the domain of $j$ is all dogs. Translate the following statements into English and find their truth value.

(a) $\forall j F(j)$
(b) $\exists j (G(j) \rightarrow H(j))$
(c) $\forall j ((F(j) \lor H(j)) \rightarrow \neg G(j))$
(d) $\exists j (\neg F(j) \land G(j)) \lor \forall j H(j)$

Solution. Let $F(j)$ be “$j$ is a good dog,” $G(j)$ be “$j$ barks wildly at street-cleaning trucks,” and $H(j)$ be “$j$ does not like walks.”

(a) Every dog is a good dog. True.
(b) There is a dog that either does not like walks or does not bark wildly at street-cleaning trucks. True (probably). (This is using the logical equivalence $p \rightarrow q \equiv \neg p \lor q$.)
(c) Dogs either do not bark wildly at street-cleaning trucks or are bad dogs who like walks. False. (Using De Morgan’s laws and $p \rightarrow q \equiv \neg p \lor q$).
(d) Either there is a bad dog who barks wildly at street-cleaning trucks or no dogs like walks. False because all dogs are good dogs and at least one dog likes walks.

3. Let $S(x, y)$ be the statement “$x$ has asked $y$ a question” and let $P(x)$ be the statement “$x$ is a student,” where the domain is all Math 55 GSIs and students and Professor Williams. Translate the following statements into symbols.

(a) Professor Williams has asked Melissa a question.
(b) Professor Williams has asked everyone a question.
(c) At least one student has asked another student a question.
(d) Nobody has asked every student a question.
(e) There is a student who has never been asked a question.

Solution. (a) $S(\text{Professor Williams, Melissa})$
(b) $\forall y S(\text{Professor Williams, } y)$
(c) $\exists x \exists y (P(x) \land P(y) \land S(x, y))$
(d) $\neg \exists x (\forall y (P(y) \land S(x, y)))$
(e) \( \exists y (P(y) \land \forall x \neg S(x, y)) \) or, equivalently, \( \exists y (P(y) \land \neg \exists x S(x, y)) \)

4. Show that \( \forall x P(x) \lor \forall x Q(x) \) and \( \forall x (P(x) \lor Q(x)) \) are not logically equivalent.

Solution. To show that these statements are not logically equivalent, we need to find a domain for the two statements such that they have different truth values on that domain.

Let the domain consist of two elements, \( j \) and \( k \) such that \( P(j) \) is true, \( P(k) \) is false, \( Q(j) \) is false, and \( Q(k) \) is true. Then the statement \( \forall x (P(x) \lor Q(x)) \) is true. However, \( j \) is a counterexample to \( \forall x P(x) \) and \( k \) is a counterexample to \( \forall x Q(x) \) so the disjunction \( \forall x P(x) \lor \forall x Q(x) \) is false.

5. Express the negations of each of these statements so that all negation symbols are in front of predicates. (Hint: remember De Morgan’s Laws)

(a) \( \neg (\forall x \exists y P(x, y) \land \forall x \exists y Q(x, y)) \)

Solution.
\[
\neg (\forall x \exists y P(x, y) \land \forall x \exists y Q(x, y)) \equiv \neg (\forall x \exists y P(x, y)) \lor \neg (\forall x \exists y Q(x, y)) \\
\equiv \exists x (\neg \exists y P(x, y)) \lor \exists x (\neg \exists y Q(x, y)) \\
\equiv \exists x \forall y \neg P(x, y) \lor \exists x \forall y \neg Q(x, y)
\]

(b) \( \neg \exists y (S(y) \lor \forall x \neg R(x, y)) \)

Solution.
\[
\neg \exists y (S(y) \lor \forall x \neg R(x, y)) \equiv \forall y (\neg (S(y) \lor \forall x \neg R(x, y)) \\
\equiv \forall y (\neg S(y) \land \neg \forall x \neg R(x, y)) \\
\equiv \forall y (\neg S(y) \land \exists x R(x, y))
\]

6. Consider the argument “If \( n \) is a real number such that \( n > 1 \), then \( n^2 > 1 \). Suppose that \( n^2 > 1 \). Then \( n > 1 \).” Is it valid? What if the last two sentences were instead “Suppose that \( n^2 < 1 \). Then \( n < 1 \).”?

Solution. The first version is not a valid argument; it is the fallacy that \( ((p \rightarrow q) \land q) \rightarrow p \), which is not a tautology. The second version is a valid argument; it is \textit{modus tollens}.