Problem Solutions

1. Choose 3 propositions p, q, and r and write (as naturally as possible): $p \wedge \neg q$, the converse of $p \to r$, the contrapositive of $\neg q \to r$, $(p \vee q) \wedge r$.

Solution. We'll use "I dropped my ice cream" for p, "I am unhappy" for q, and "I like chocolate cake" for r.

- $(p \wedge \neg q)$ I dropped my ice cream and I am happy.
- (Converse of $p \to \neg r$) I don't like chocolate cake, so I dropped my ice cream.
- (Contrapositive of $\neg q \to r$) If I do not like chocolate cake, then I am unhappy.
- $((p \wedge \neg r) \leftrightarrow q)$ I dropped my ice cream and I don't like chocolate cake, so I am unhappy (and vice-versa).

2. Can the conditional statement $p \to q$ be true if q is false? What about if p is false? If so, give an example.

Solution. Yes to both. If q is false, then $p \to q$ is true when p is false. Example: "If monkeys are purple, then the moon is made of white chocolate." If p is false, then $p \to q$ is true for any truth value of q. Example: "If the moon is made of white chocolate, then 7 divides 14."

Aside: if p is false, the conditional statement $p \to q$ is called "vacuously true."

3. Show that a conditional statement and its contrapositive are logically equivalent.

Solution. Consider the conditional statement $p \to q$.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

4. Give an example of a tautology and a contradiction (written in symbols).

Solution. $p \vee \neg p$ is a tautology. $p \wedge \neg p$ is a contradiction.

5. (De Morgan's Laws) Convince yourself that $\neg(p \wedge q) \equiv \neg p \vee \neg q$ by

(a) choosing 2 propositions (whose truth value you know) for p and q and determining the truth value of $\neg(p \wedge q)$ and $\neg p \vee \neg q$

(b) using a truth table.

Is (a) alone enough to show that the two propositions are logically equivalent?

Solution. (a) Using "The moon is made of white chocolate" for p and "7 divides 14" as q, we have that $\neg(p \wedge q)$ is "It is not the case that both the moon is made of white chocolate and 7 divides 14" and $\neg p \vee \neg q$ is "Either the moon is not made of white chocolate or 7 does not divide 14." Both of these propositions are true.

(b)

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

No, (a) alone is not enough, because we have only verified that the truth values of the two propositions are the same for a single choice of truth value for p and q.

6. Show that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

Solution.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T