9.5: Linear 1st Order Diff Eqns Problem Solutions

4/10/17

Find the solution of the DEs.

1. $xy' - 2y = x^2 \quad (x>0)$

Soln: While this is a first order linear DE, it is not in the correct form, so we divide through by x:

$$xy' - 2y = x^2$$

$$y' - \frac{2}{x}y = x$$

$P(x) = -\frac{2}{x}$

$\int P(x)\,dx = -2\ln(x)$

$e^{\int P(x)dx} = e^{-2\ln(x)} = e^{\ln(x^{-2})}$ using ln rules

$= x^{-2}$

So the integrating factor is

Multiplying through by the integrating factor gives

$$\underbrace{y'x^{-2} - 2x^{-3}y}_{\frac{d}{dx}(yx^{-2})} = x^{-1}$$

$$\frac{d}{dx}(yx^{-2}) = x^{-1}$$

Integrating both sides: $\int \frac{d}{dx}(yx^{-2})\,dx = \int x^{-1}\,dx$

$$yx^{-2} = \ln(x) + C$$

$$y = x^2\ln(x) + Cx^2$$

2. $xy' + y = x\ln x$

Soln: Again, we divide through by x to put the DE in the correct form

$$xy' + y = x\ln x$$

$$y' + \frac{1}{x}y = \ln x$$

$P(x) = \frac{1}{x}$, so $\int P(x)\,dx = \ln x$

$e^{\int P(x)dx} = e^{\ln x} = x$, so the integrating factor is x.

Multiplying through by the integrating factor gives

$$\underbrace{xy' + y}_{\frac{d}{dx}(xy)} = x\ln x$$

(Notice this is the equation we started with; we could have saved ourselves some work by recognizing the LHS as $\frac{d}{dx}(xy)$)

$$\frac{d}{dx}(xy) = x\ln x$$

Integrating both sides gives

$$xy = \int x\ln x\,dx$$

To evaluate $\int x \ln x \, dx$, we integrate by parts.

$u = \ln x \quad dv = x$

$du = \frac{1}{x} \quad v = \frac{x^2}{2}$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

So $xy = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

$y = \frac{x}{2} \ln x - \frac{x}{4} + \frac{C}{x}$.