Midterm 2 Review

3/17/17

These problems are from practice midterm 6 on the course webpage.

1. True/False. If true, justify. If false, give a counterexample.

a) If $|a_n| \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Answer: False. $a_n = \frac{1}{n}$ is a counterexample; $|a_n| = \left|\frac{1}{n}\right| \to 0$ as $n \to \infty$, but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

b) The following series converges: $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2n}{n!}$

Answer: False. $\frac{2 \cdot 4 \cdot 6 \cdots 2n}{n!} \overset{(2\cdot1)\cdot(2\cdot2)\cdot(2\cdot3)\cdots(2\cdot n)}{=} \frac{2^n\, n!}{n!} = 2^n$, and $\lim_{n\to\infty} 2^n = \infty$, so the series diverges by the divergence test. (You could also use the ratio test here)

2. The series $\sum_{n=0}^{\infty} c_n (x-2)^n$ converges at $x=-2$ & diverges at $x=8$. What can we conclude (i.e. converges, diverges, or insufficient information) about:

a) $\sum_{n=0}^{\infty} c_n$ b) $\sum_{n=0}^{\infty} c_n (-3)^n$ c) $\sum_{n=0}^{\infty} c_n 10^n$

Answer: First, notice that the series is centered at 2. Since it converges at $x=-2$, we know the radius of convergence is at least $|2-(-2)| = 4$. Since it diverges at 8, we know the radius of convergence is at least $|2-8| = 6$.

a) Here, $x=3$. $|2-3| = 1 < 4$ ← smallest possible radius of convergence, so this series converges.

b) Here, $x=-1$. $|2-(-1)| = 3 < 4$, so this series converges.

c) Here, $x=12$. $|2-12| = 10 > 6$ ← largest possible radius of convergence, so the series diverges.

3. Find the Maclaurin series of $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$. Find the radius of convergence of the series as well.

Answer: The Maclaurin series of $\sin x$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, so the Maclaurin series of $\frac{\sin x}{x} = \frac{1}{x} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}\, x^{-1}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

To find the radius of convergence,

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}\,x^{2(n+1)}}{(2(n+1)+1)!}}{\dfrac{(-1)^n\,x^{2n}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{x^{2n}}\cdot\frac{(2n+1)!}{(2n+3)!}\right|$$

$$= |x^2|\lim_{n\to\infty}\left|\frac{1}{(2n+2)(2n+3)}\right| = 0 < 1,$$ so radius of convergence is ∞.

4. Is the following series convergent or divergent?

$$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{1}{n^3}\right)}{\sin\left(\frac{1}{n}\right)}.$$

<u>Answer:</u> Since this series doesn't look like any of the series we know well, we first just try to get a sense of what the terms approach as $n\to\infty$.

The Maclauren series of $\sin(u)$ is $\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$

So letting $u = \frac{1}{x^3}$, $\sin\left(\frac{1}{x^3}\right) = \frac{1}{x^3} - \frac{\left(\frac{1}{x^3}\right)^3}{3!} + \frac{\left(\frac{1}{x^3}\right)^5}{5!} - \cdots$

$$= \frac{1}{x^3} - \frac{1}{3!\,x^6} + \frac{1}{5!\,x^8} - \cdots$$

As $x\to\infty$, the dominant term is $\frac{1}{x^3}$, since other terms are much smaller.

So $\sin\left(\frac{1}{x^3}\right) \to \frac{1}{x^3}$ as $x\to\infty$.

Similarly, $\sin\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{\left(\frac{1}{x}\right)^3}{3!} + \frac{\left(\frac{1}{x}\right)^5}{5!} - \cdots$, so $\sin\left(\frac{1}{x}\right) \to \frac{1}{x}$ as $x\to\infty$.

$$\frac{\sin\left(\frac{1}{n^3}\right)}{\sin\left(\frac{1}{n}\right)} \to \frac{\frac{1}{n^3}}{\frac{1}{n}} = \frac{1}{n^2}$$ as $n\to\infty$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we guess our series converges. We use limit comparison with $\frac{1}{n^2}$. (It's easiest if we write $\frac{1}{n^2} = \frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n}\right)}$)

$$\lim_{n\to\infty}\frac{\left(\dfrac{\sin\left(\frac{1}{n^3}\right)}{\sin\left(\frac{1}{n}\right)}\right)}{\left(\dfrac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n}\right)}\right)} = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n^3}\right)}{\sin\left(\frac{1}{n}\right)}\cdot\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}}\cdot\frac{\frac{1}{n}}{\sin\left(\frac{1}{n}\right)}$$

$$= 1 \text{ since } \lim_{x\to\infty}\frac{\sin(x^{-3})}{x^{-3}} \leftarrow \text{"}\tfrac{0}{0}\text{"}$$

$$\overset{l'H}{=} \lim_{x\to\infty}\frac{\cos(x^{-3})\cdot -3x^{-4}}{-3x^{-4}}$$

$$= 1$$

and $\lim_{x\to\infty}\frac{\sin(x^{-1})}{x^{-1}} \leftarrow \text{"}\tfrac{0}{0}\text{"}$

$$\overset{l'H}{=} \lim_{x\to\infty}\frac{\cos(x^{-1})(-x^{-2})}{(-x^{-2})} = 1.$$

So, by the limit comparison test, the series converges.

5. Find a power series representing $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ & determine the radius & interval of convergence.

<u>Answer 1:</u> (If you remember the Maclaurin series of $\ln(1+x)$) First, we rewrite using log rules

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).$$

Now, $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ (for $|x|<1$)

So $\ln(1-x) = \ln(1-(-x)) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-x)^n}{n}$ (also for $\underset{|x|}{\underset{\shortparallel}{|-x|}} < 1$)

$$= \sum_{n=1}^{\infty} (-1)^{n-1} (-1)^n \frac{x^n}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{x^n}{n}$$

$$= -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Then $f(x) = \ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} - \left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right)$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$= \sum_{n=1}^{\infty} \left((-1)^{n-1} + 1\right) \frac{x^n}{n}$$

$$= \sum_{n=0}^{\infty} 2 \frac{x^{2n+1}}{2n+1}$$

If n is even, then $n-1$ is odd, so $(-1)^{n-1} = -1$ & $(-1)^{n-1} + 1 = 0$
If n is odd, then $n-1$ is even, so $(-1)^{n-1} = 1$ & $(-1)^{n-1} + 1 = 2$

The radius & interval of convergence of this series is the same as for (you can check this using ratio test) $\ln(1+x)$).

<u>Answer 2:</u> (If you don't remember the Maclaurin series of $\ln(1+x)$).

We rewrite using log rules first.

$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$. We don't want to use the Taylor series formula to find a power series representation of this function (it would take too long; it is a valid way to do this problem, though), so we need to use something else. We notice that $f'(x) = \frac{1}{1+x} + \frac{1}{1-x}$, and we know power series representations of these two functions.

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (for $|x|<1$) and $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$

$$= \sum_{n=0}^{\infty} (-1)^n x^n \quad (\text{for } |x|<1).$$

So $f'(x) = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x|<1$

$= \sum_{n=0}^{\infty} (1+(-1)^n) x^n$

If n is odd, $(-1)^n = -1$, so $1+(-1)^n = 0$.
If n is even, $(-1)^n = 1$, so $1+(-1)^n = 2$. So all odd-exponent terms of this sum are zero, & all even exponent terms are $2x^{2n}$

$= \sum_{n=0}^{\infty} 2x^{2n}$

$f(x) = \int f'(x)\,dx = \sum_{n=0}^{\infty} 2 \int x^{2n}\,dx$

$= \sum_{n=0}^{\infty} 2 \frac{x^{2n+1}}{2n+1} + C$ & interval of convergence is the same as the interval of convergence of, which is $(-1,1)$.

To find C, we remember $f(x) = \ln\left(\frac{1+x}{1-x}\right)$.

So $\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} 2 \frac{x^{2n+1}}{2n+1} + C$

If $x=0$,

$\underset{\substack{\parallel \\ 0}}{\ln\left(\frac{1}{1}\right)} = C$, so $C=0$ & $\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$.

6. Determine ~~the~~ if the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}+1}$ diverges, converges absolutely, or converges conditionally.

<u>Answer:</u> We start by testing for absolute convergence. We consider the series

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$.

As $n \to \infty$, $\frac{1}{\sqrt{n}+1} \to \frac{1}{\sqrt{n}}$, so we guess divergence. Using limit comparison with $\frac{1}{\sqrt{n}}$, (since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, as it is a p-series with $p = \frac{1}{2} < 1$)

$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = 1$. So the "absolute value" series of the series we're interested in diverges, which means $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}+1}$ is NOT absolutely convergent.

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}+1}$ converges by alternating series test

($\lim_{n\to\infty} \frac{1}{\sqrt{n}+1} = 0$ and $\sqrt{n+1}+1 \geq \sqrt{n}+1$, so $\frac{1}{\sqrt{n+1}+1} \leq \frac{1}{\sqrt{n}}$).

Thus, the series is conditionally convergent.

7. Determine if the sequence converges or diverges.

$a_n = (-1)^n \sqrt{n}\left(\ln(n+1) - \ln(n)\right)$.

<u>Answer:</u> First, we rewrite $a_n = (-1)^n \sqrt{n} \ln\left(\frac{n+1}{n}\right)$

$= (-1)^n \sqrt{n} \ln\left(1 + \frac{1}{n}\right)$

This converges only if $|a_n|$ goes to zero, since otherwise, if $|a_n|$ goes to L, a_n would "bounce" between L & $-L$.

So we consider $|a_n| = \sqrt{n} \ln\left(1 + \frac{1}{n}\right)$.

$\lim_{n\to\infty} \sqrt{n} \ln\left(1+\frac{1}{n}\right)$ is of the form "$\infty \cdot 0$", so we switch to functions and try to use l'Hospital's.

$$\lim_{x\to\infty} \sqrt{x} \ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{\sqrt{x}}} \leftarrow \text{"}\frac{0}{0}\text{"}$$

$$\overset{l'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot -x^{-2}}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to\infty} -2\left(\frac{-x^{-1/2}}{1+\frac{1}{x}}\right)$$

$$= \lim_{x\to\infty} 2\left(\frac{1}{\sqrt{x}\left(1+\frac{1}{x}\right)}\right) = \lim_{x\to\infty} \frac{2}{\sqrt{x} + \frac{1}{\sqrt{x}}} = 0.$$

So $|a_n| \to 0$, which implies $a_n \to 0$ as $n \to \infty$.