These problems are from practice midterm 6 on the course webpage.

1. True/False. If true, justify. If false, give a counterexample.
   a) If \(|a_n| \to 0\) as \(n \to \infty\), then \(\sum_{n=1}^{\infty} a_n\) converges absolutely.
   Answer: False. \(a_n = \frac{1}{n}\) is a counterexample; \(|a_n| = \frac{1}{n}\) \(\to 0\) as \(n \to \infty\), but \(\sum_{n=1}^{\infty} \frac{1}{n}\) does not converge.

   b) The following series converges: \(\sum_{n=1}^{\infty} \frac{2, 4, 6, \ldots, 2n}{n!}\)
   Answer: False. \(\frac{2, 4, 6, \ldots, 2n}{n!} = \frac{2^n n!}{n!} = 2^n\), and \(\lim_{n \to \infty} 2^n = \infty\), so the series diverges by the divergence test. (You could also use the ratio test here.)

2. The series \(\sum_{n=0}^{\infty} c_n (x-2)^n\) converges at \(x=2\) & diverges at \(x=8\). What can we conclude (i.e. converges, diverges, or insufficient information) about:
   a) \(\sum_{n=0}^{\infty} c_n\)
   b) \(\sum_{n=0}^{\infty} c_n (-3)^{n}\)
   c) \(\sum_{n=0}^{\infty} c_n 10^{n}\)

   Answer: First, notice that the series is centered at 2. Since it converges at \(x=2\), we know the radius of convergence is at least \(|2-(-2)| = 4\). Since it diverges at \(8\), we know the radius of convergence is at least \(|8-2|=6\).
   a) Here, \(x=3\). \(12-3| = 12-3| = 12-3\cdot6 = 12-18 = -6\), so this series diverges.
   b) Here, \(x=-1\). \(12-(-1)| = 12-1 = 11\), so this series converges.
   c) Here, \(x=12\). \(12-12| = 12-12 = 0\), largest possible radius of convergence, so the series diverges.

3. Find the Maclaurin series of \(f(x) = \begin{cases} \sin x & x \neq 0 \\ 1 & x = 0 \end{cases}\). Find the radius of convergence of the series as well.

   Answer: The Maclaurin series of \(\sin x\) is \(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\), so the Maclaurin series of \(\frac{\sin x}{x}\) is \(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\).
To find the radius of convergence,

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2n+1)!} \right| = \lim_{n \to \infty} \left| \frac{x^{2n+2} (2n+1)!}{x^n (2n1)!} \right| = 1^2 \lim_{n \to \infty} \left| \frac{1}{(2n+2)(2n+3)} \right| = 0 < 1,
\]
so the radius of convergence is $\infty$.

4. Is the following series convergent or divergent?

\[\sum_{n=1}^{\infty} \frac{\sin \left(\frac{1}{n}\right)}{n} \]

**Answer:** Since this series doesn't look like any of the series we know well, we first just try to get a sense of what the terms approach as $n \to \infty$.

The Maclaurin series for $\sin(u)$ is $\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \ldots$

So letting $u = \frac{1}{x^3}$, $\sin \left(\frac{1}{x^3}\right) = \frac{1}{x^3} - \frac{(\frac{1}{x^3})^3}{3!} + \frac{(\frac{1}{x^3})^5}{5!} - \ldots$

or

\[\frac{1}{x^3} - \frac{1}{3! \cdot x^6} + \frac{1}{5! \cdot x^8} - \ldots \]

As $x \to \infty$, the dominant term is $\frac{1}{x^3}$, since other terms are much smaller.

So $\sin \left(\frac{1}{x^3}\right) \to \frac{1}{x^3}$ as $x \to \infty$.

Similarly, $\sin \left(\frac{1}{n}\right) = \frac{1}{n} - \frac{(\frac{1}{n})^3}{3!} + \frac{(\frac{1}{n})^5}{5!} - \ldots$, so $\sin \left(\frac{1}{n}\right) \to \frac{1}{n}$ as $n \to \infty$.

Our series converges. We use limit comparison with $\frac{1}{n^2}$: (It's easiest if we write $\frac{1}{n} = \frac{1}{n^3} \cdot \sin \left(\frac{1}{n}\right)$.)

\[
\lim_{n \to \infty} \left( \frac{\sin \left(\frac{1}{n}\right)}{\sin \left(\frac{1}{n^3}\right)} \right) = \lim_{n \to \infty} \frac{\frac{1}{n^3} \cdot \sin \left(\frac{1}{n^3}\right)}{\sin \left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{\sin \left(\frac{1}{n^3}\right)}{\frac{1}{n^3}} \cdot \frac{\frac{1}{n^3}}{\sin \left(\frac{1}{n}\right)} = 1 \text{ since } \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \text{ and } \lim_{x \to 0} \frac{x}{\sin(x)} = 1 \text{ as } x \to 0.\]

4/4 = \lim_{n \to \infty} \frac{\cos \left(\frac{1}{n^3}\right) \cdot 3x^4}{-3x^4} = 1.

So, by the limit comparison test, the series converges.
5. Find a power series representing \( f(x) = \ln\left(\frac{1+x}{1-x}\right) \) & determine the radius & interval of convergence.

Answer 1: First, we rewrite using log rules

\[
f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).
\]

Now, \( \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \) (for \( |x| < 1 \))

So \( \ln(1-x) = \ln(1-(1-x)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (1-x)^n}{n} \) (also for \( |1-x| < 1 \))

\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}
\]

\[
= \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{x^n}{n}
\]

\[
= -\sum_{n=1}^{\infty} \frac{x^n}{n}
\]

Then \( f(x) = \ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} - \left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right) \)

\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n}
\]

\[
= \sum_{n=1}^{\infty} (-1)^{n-1+1} \frac{x^n}{n}
\]

\[
= \sum_{n=1}^{\infty} 2 \frac{x^{2n+1}}{2n+1}
\]

The radius & interval of convergence of this series is the same as for \( \ln(1+x) \) (you can check this using the ratio test).

Answer 2: If you don’t remember the Maclaurin series of \( \ln(1+x) \),

We rewrite using log rules first.

\( f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) \). We don’t want to use the Taylor series formula to find a power series representation of this function (it would take too long; it is a valid way to do this problem, though), so we need to use something else. We notice that \( f'(x) = \frac{1}{1+x} + \frac{1}{1-x} \), and we know power series representations of these two functions.

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{(for \( |x| < 1 \))}
\]

\[
\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n \quad \text{(for \( |x| < 1 \))}
\]
\[ f'(x) = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{for} \ |x| < 1 \]

If \( n \) is odd, \((-1)^n = -1\), so \(1 + (-1)^n = 0\).
If \( n \) is even, \((-1)^n = 1\), so \(1 + (-1)^n = 2\). So all odd-exponent terms of this sum are zero, & all even-exponent terms are \(2x^{2n}\).

\[ f(x) = \int f'(x) \, dx = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} + C \text{ & interval of convergence is the same as the interval of convergence of, which is } (-1, 1). \]

To find \( C \), we remember \( f(x) = \ln\left(\frac{1+x}{1-x}\right) \).

So \( \ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} + C \)

If \( x = 0 \), \( \ln(1) = C \), so \( C = 0 \) & \( \ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} \).

6. Determine if the series \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n+1}} \) diverges, converges absolutely, or converges conditionally.

**Answer**: We start by testing for absolute convergence. We consider the series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \).

As \( n \to \infty \), \( \frac{1}{\sqrt{n+1}} \to \frac{1}{\sqrt{n}} \), so we guess divergence. Using limit comparison, \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges, since \( \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \) diverges.

With \( \frac{1}{\sqrt{n}} \), \( \lim_{n \to \infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{1}{n} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1 \). So the "absolute value" series of the series we're interested in diverges, which means \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n+1}} \) is not absolutely convergent.

\( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n+1}} \) converges by alternating series test (\( \lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0 \) and \( \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \)).

Thus, the series is conditionally convergent.
7. Determine if the sequence converges or diverges.

\[ a_n = (-1)^n \sqrt[n]{\ln(n+1) - \ln(n)} \]

**Answer:** First, we rewrite \[ a_n = (-1)^n \sqrt[n]{\ln\left(\frac{n+1}{n}\right)} \]

\[ = (-1)^n \sqrt[n]{\ln(1 + \frac{1}{n})} \]

This converges only if \( |a_n| \to 0 \), since otherwise, if \( |a_n| \to L \), an would "bounce" between \( L \) and \(-L\).

So we consider \( |a_n| = \sqrt[n]{\ln(1 + \frac{1}{n})} \).

\[ \lim_{n \to \infty} \sqrt[n]{\ln(1 + \frac{1}{n})} \] is of the form \( \infty \cdot 0 \), so we switch to functions and try to use L'Hôpital's.

\[ \lim_{x \to \infty} \sqrt{x} \ln(1 + \frac{1}{x}) = \lim_{x \to \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{\sqrt{x}}} \]

\[ = \lim_{x \to \infty} \frac{1}{1 + \frac{3}{2} x^{3/2}} - x^{-2} \]

\[ = \lim_{x \to \infty} -2 \left( \frac{-x^{-1/2}}{1 + \frac{1}{x}} \right) \]

\[ = \lim_{x \to \infty} 2 \left( \frac{1}{\sqrt{x}(1 + \frac{1}{x})} \right) = \lim_{x \to \infty} \frac{2}{\sqrt{x} + \frac{1}{\sqrt{x}}} = 0. \]

So \( |a_n| \to 0 \), which implies \( a_n \to 0 \) as \( n \to \infty \).