11.4: Comparison Tests Problem Solutions

Determine if the series converges.

1. $\sum_{n=3}^{\infty} \frac{n^2-n-1}{n^4+5}$

Soln: When $n \to \infty$, $\frac{n^2-n-1}{n^4+5} \to \frac{n^2}{n^4} = \frac{1}{n^2}$, so we expect convergence. We try comparison test & our goal is to get that $\frac{n^2-n-1}{n^4+5}$ is less than some function that looks like $\frac{1}{n^2}$.

$$\frac{n^2-n-1}{n^4+5} = \frac{n^2}{n^4+5} - \frac{n+1}{n^4+5} < \frac{n^2}{n^4+5} < \frac{n^2}{n^4} \quad (\text{since } n^4 < n^4+5)$$
$$= \frac{1}{n^2}.$$

$\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges (p-series with $p>1$), so the original series converges.

Note: Limit comparison with $a_n = \frac{n^2-n-1}{n^4+5}$ & $b_n = \frac{1}{n^2}$ also works.

2. $\sum_{n=2}^{\infty} \frac{n}{n^5-1}$

Soln: When $n \to \infty$, $\frac{n}{n^5-1} \to \frac{n}{n^5} = \frac{1}{n^4}$, and $\sum_{n=2}^{\infty} \frac{1}{n^4}$ converges, so we expect convergence. Since $\frac{n}{n^5-1} > \frac{n}{n^5}$ (the wrong inequality to be useful for comparison test), we do limit comparison with $a_n = \frac{n}{n^5-1}$, $b_n = \frac{1}{n^4}$.

$$\lim_{n\to\infty} \frac{\left(\frac{n}{n^5-1}\right)}{\left(\frac{1}{n^4}\right)} = \lim_{n\to\infty} \frac{n^5}{n^5-1} \cdot \frac{\frac{1}{n^5}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^5}} = 1,$$ so $\sum_{n=2}^{\infty} a_n$ converges as long as $\sum_{n=2}^{\infty} b_n$ converges. $\sum_{n=2}^{\infty} b_n$ converges, so our series does as well.

3. $\sum_{n=1}^{\infty} \frac{6^n}{4^n+3^n}$

Soln: We first rewrite the terms of this series: $\frac{6^n}{4^n+3^n} = \frac{6^n}{4^n\left(1+\left(\frac{3}{4}\right)^n\right)}$

Since $\left(\frac{3}{4}\right)^n \to 0$ as $n \to \infty$, this sequence looks like $\frac{6^n}{4^n}$ when n is large

$\sum_{n=1}^{\infty} \left(\frac{6}{4}\right)^n$ diverges, so we guess that this diverges.

Comparison test seems difficult (hard to find a lower bound) so we use limit comparison with $b_n = \left(\frac{6}{4}\right)^n$.

$$\lim_{n\to\infty} \frac{\left(\frac{6^n}{4^n+3^n}\right)}{\left(\frac{6^n}{4^n}\right)} = \lim_{n\to\infty} \frac{6^n}{4^n+3^n} \cdot \frac{4^n}{6^n}$$

$$= \lim_{n\to\infty} \frac{4^n}{4^n+3^n} \cdot \frac{\frac{1}{4^n}}{\frac{1}{4^n}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\left(\frac{3}{4}\right)^n} = 1.$$

$\sum_{n=1}^{\infty} \left(\frac{6}{4}\right)^n$ diverges, so this implies $\sum_{n=1}^{\infty} \frac{6^n}{4^n+3^n}$ diverges as well.

4. $\sum_{n=1}^{\infty} \frac{1+\sin^2 n}{e^n}$

Soln: Notice $0 \le \sin^2 n \le 1$, so $\frac{\sin^2 n}{e^n} \le \frac{1}{e^n} = \left(\frac{1}{e}\right)^n$.

$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series & $\frac{1}{e} < 1$, so it converges. By comparison test, $\sum_{n=1}^{\infty} \frac{1+\sin^2 n}{e^n}$ also converges.

5. $\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$.

Soln: As $n \to \infty$, $\frac{n+3^n}{n+2^n}$ looks like $\left(\frac{3}{2}\right)^n$, which goes to infinity, so we try the divergence test.

$$\lim_{n\to\infty} \frac{n+3^n}{n+2^n} = \lim_{x\to\infty} \frac{x+3^x}{x+2^x}$$

(since $\frac{x+3^x}{x+2^x}$ doesn't "bounce", even if this limit is infinite, it will equal the limit of the sequence)

$$= \lim_{x\to\infty} \frac{1+(\ln 3)3^x}{1+(\ln 2)2^x}$$

(use l'Hospital's, since limit is of the form "$\frac{\infty}{\infty}$")

$$= \lim_{x\to\infty} \frac{(\ln 3)^2 3^x}{(\ln 2)^2 2^x}$$

$$= \left(\frac{\ln 3}{\ln 2}\right)^2 \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty.$$

So this series doesn't converge.

6. $\sum_{n=1}^{\infty} \frac{n+2^n}{n+3^n}$.

Soln: As $n \to \infty$, $\frac{n+2^n}{n+3^n}$ looks like $\frac{2^n}{3^n}$; $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ converges (geometric

series & $\frac{2}{3} < 1$), so we expect convergence.

We try limit comparison with $b_n = \left(\frac{2}{3}\right)^n$ (using the comparison test is possible, but more complicated).

$$\lim_{n\to\infty} \frac{\left(\frac{n+2^n}{n+3^n}\right)}{\left(\frac{2^n}{3^n}\right)} = \lim_{n\to\infty} \frac{n+2^n}{n+3^n} \cdot \frac{3^n}{2^n}$$

$$= \lim_{n\to\infty} \frac{n+2^n}{n+3^n} \cdot \frac{\left(\frac{1}{2^n}\right)}{\left(\frac{1}{3^n}\right)}$$

$$= \lim_{n\to\infty} \frac{\frac{n}{2^n}+1}{\frac{n}{3^n}+1}$$

$$= 1.$$

Note: $\lim_{x\to\infty} \frac{x}{2^x}$ is "$\frac{\infty}{\infty}$"

$$\overset{l'H}{=} \lim_{x\to\infty} \frac{1}{\ln(2)2^x} = 0$$

So $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{x\to\infty} \frac{x}{2^x} = 0$

Similarly, $\lim_{n\to\infty} \frac{n}{3^n} = 0$.

Since $\sum_{n=1}^{\infty} b_n$ converges, so does our series.