Determine if the series converges.

1. \( \sum_{n=3}^{\infty} \frac{n^2-n-1}{n^4+5} \)

**Soln:** When \( n \to \infty \), \( \frac{n^2-n-1}{n^4+5} \to \frac{n^2}{n^4} = \frac{1}{n^2} \), so we expect convergence.

We try comparison test & our goal is to get that \( \frac{n^2-n-1}{n^4+5} \) is less than some function that looks like \( \frac{1}{n^2} \).

\[
\frac{n^2-n-1}{n^4+5} = \frac{n^2}{n^4+5} - \frac{n+1}{n^4+5} \leq \frac{n^2}{n^4+5} \leq \frac{n^2}{n^4} = \frac{1}{n^2}.
\]

\( \sum_{n=3}^{\infty} \frac{1}{n^2} \) converges (p-series with \( p > 1 \)), so the original series converges.

**Note:** Limit comparison with \( a_n = \frac{n^2-n-1}{n^4+5} \) & \( b_n = \frac{1}{n^2} \) also works.

2. \( \sum_{n=2}^{\infty} \frac{n}{n^5-1} \)

**Soln:** When \( n \to \infty \), \( \frac{n}{n^5-1} \to \frac{1}{n^4} \), and \( \sum_{n=2}^{\infty} \frac{1}{n^4} \) converges, so we expect convergence. Since \( \frac{n}{n^5-1} < \frac{n}{n^5} \) (the wrong inequality to be useful for comparison test), we do limit comparison with \( a_n = \frac{n}{n^5-1} \), \( b_n = \frac{1}{n^4} \).

\[
\lim_{n \to \infty} \left( \frac{\frac{n}{n^5-1}}{\frac{1}{n^4}} \right) = \lim_{n \to \infty} \frac{n}{n^5} \cdot \frac{n^4}{1} = \lim_{n \to \infty} \frac{1}{1 - \frac{1}{n^5}} = 1,
\]

so \( \sum_{n=2}^{\infty} a_n \) converges as long as \( \sum_{n=2}^{\infty} b_n \) converges. \( \sum_{n=2}^{\infty} b_n \) converges, so our series does as well.

3. \( \sum_{n=1}^{\infty} \frac{6^n}{4^n+3^n} \)

**Soln:** We first rewrite the terms of this series: \( \frac{6^n}{4^n+3^n} = \frac{6^n}{4^n(1+(\frac{3}{4})^n)} \)

Since \( (\frac{3}{4})^n \to 0 \) as \( n \to \infty \), this looks like \( \frac{6^n}{4^n} \) when \( n \) is large.

\( \sum_{n=1}^{\infty} \frac{6^n}{4^n} \) diverges, so we guess that this diverges.
Comparison test seems difficult (hard to find a lower bound) so we use limit comparison with \( b_n = (\frac{6}{x})^n \).

\[
\lim_{n \to \infty} \frac{(4^n+3^n)}{(6^n)} = \lim_{n \to \infty} \frac{4^n}{6^n} = \frac{4^n}{4^n} = 1
\]

\[
\lim_{n \to \infty} \frac{1}{1 + (\frac{3}{x})^n} = 1
\]

implies \( \sum_{n=1}^{\infty} \frac{6^n}{4^n+3^n} \) diverges, so this implies \( \sum_{n=1}^{\infty} \frac{6^n}{4^n+3^n} \) diverges as well.

4. \( \sum_{n=1}^{\infty} \frac{1+\sin^2 n}{e^n} \)

**Soln:** Notice \( \sin^2 n \leq 1 \), so \( \frac{\sin^2 n}{e^n} \leq \frac{1}{e^n} = (\frac{1}{e})^n \).

\( \sum_{n=1}^{\infty} (\frac{1}{e})^n \) is a geometric series \( \Rightarrow \frac{1}{1-e} \leq 1 \), so it converges. By comparison test, \( \sum_{n=1}^{\infty} \frac{1+\sin^2 n}{e^n} \) also converges.

5. \( \sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n} \)

**Soln:** As \( n \to \infty \), \( \frac{n+3^n}{n+2^n} \) looks like \( \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \), which goes to infinity, so we try the divergence test.

\[
\lim_{n \to \infty} \frac{n+3^n}{n+2^n} = \lim_{n \to \infty} \frac{x+3^x}{x+2^x} \quad \text{(since \( \frac{x+3^x}{x+2^x} \) doesn't "bounce", even if this limit is infinite it will equal the limit of the sequence)}
\]

\( \lim_{x \to \infty} \frac{1+ (\ln 3)^x}{1+ (\ln 2)^x} \) (use l'Hospital's, since limit is of the form \( \frac{\infty}{\infty} \))

\[
\lim_{x \to \infty} \frac{(\ln 3)^2 3^x}{(\ln 2)^2 2^x} = \infty.
\]

So this series doesn't converge.

6. \( \sum_{n=1}^{\infty} \frac{n+2^n}{n+3^n} \)

**Soln:** As \( n \to \infty \), \( \frac{n+2^n}{n+3^n} \) looks like \( \frac{2^n}{3^n} \); \( \sum_{n=1}^{\infty} \frac{2^n}{3^n} \) converges (geometric
series $a \frac{2}{3} < 1$, so we expect convergence.

We try limit comparison with $b_n = (\frac{2}{3})^n$ (using the comparison test is possible, but more complicated).

$$\lim_{n \to \infty} \frac{\frac{n+2^n}{n+3^n}}{\left(\frac{2^n}{3^n}\right)} = \lim_{n \to \infty} \frac{n+2^n}{n+3^n} \cdot \frac{3^n}{2^n}$$

$$= \lim_{n \to \infty} \frac{n+2^n}{n+3^n} \cdot \left(\frac{\frac{1}{2^n}}{\frac{1}{3^n}}\right)$$

$$= \lim_{n \to \infty} \frac{n+2^n}{n+3^n} \cdot \frac{3^n}{2^n}$$

$$= \lim_{n \to \infty} \frac{n}{2^n} + 1$$

$$= 1.$$ 

Note: $\lim_{x \to \infty} \frac{x}{x} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{1}{\ln(2)2^x} = 0$

So $\lim_{n \to \infty} \frac{n}{2^n} = \lim_{x \to \infty} \frac{x}{2^x} = 0$

Similarly, $\lim_{n \to \infty} \frac{n}{3^n} = 0$.

Since $\sum b_n$ converges, so does our series.