1. What is the "form" of the partial fraction decomposition of
   a) \( \frac{x-6}{x^3+x^2-6x} \) and b) \( \frac{1}{(x^2+1)^2(x-3)^3} \)? (Don't find the constants)

   **Solu:**
   a) \( \frac{x-6}{x^3+x^2-6x} = \frac{x-6}{x(x^2+x-6)} = \frac{x-6}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2} \)

   b) \( \frac{1}{(x^2+1)^2(x-3)^3} = \frac{A_1 x+B_1}{(x^2+1)} + \frac{A_2 x+B_2}{(x^2+1)^2} + \frac{C_1}{x-3} + \frac{C_2}{(x-3)^2} + \frac{C_3}{(x-3)^3} \)

Evaluate the following integrals:

2. \( \int \frac{x-4}{x^2-5x+6} \, dx \)

   **Solu:** The rational function here is proper, so we proceed to partial fractions.
   \( \frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \)

   Multiplying both sides by \((x-2)(x-3)\) gives
   \[ x-4 = A(x-3) + B(x-2) \]
   This is true for all \(x\), so it's true if \(x=3\), in which case the equation becomes
   \[ 3-4 = A(3-3) + B(3-2) \]
   \[ 3-4 = -A \cdot 0 + B \cdot 1 \]
   \[ -1 = B \]

   If \(x=2\), we have
   \[ 2-4 = A(2-3) + B(2-2) \]
   \[ -2 = -A \cdot 1 + B \cdot 0 \]
   \[ -2 = -A \]
   \[ A = 2 \]

   Then \( \int \frac{x-4}{x^2-5x+6} \, dx = \int \frac{2}{x-2} \, dx - \frac{1}{x-3} \, dx \)

   \[
   = 2 \ln |x-2| - \ln |x-3| \bigg|_0^1
   = 2 \left[ \ln(1) - \ln(2) \right] - \left[ \ln(2) - \ln(3) \right]
   = 2 \ln(\frac{1}{2}) - \ln(\frac{2}{3}) = \ln(\frac{1}{4}) - \ln(\frac{2}{3}) = \ln(\frac{1}{3}) = \ln(\frac{3}{8}).
   \]
3. \[ \int \frac{x^3 + 4x^2 + x - 1}{x^3 + x^2} \, dx \]

**Sln:** The degree of the numerator is equal to the degree of the denominator, so we first (polynomial) long divide.

\[
\frac{x^3 + 4x^2 + x - 1}{x^3 + x^2} = \frac{x^3 + x^2 + 3x^2 + x - 1}{x^3 + x^2} = \frac{x^3 + x^2}{x^3 + x^2} + \frac{3x^2 + x - 1}{x^3 + x^2} = 1 + \frac{3x^2 + x - 1}{x^3 + x^2}
\]

Now, we do partial fractions on (\(\ast\)), since it's a "proper" rational fn.

\[
\frac{3x^2 + x - 1}{x^3 + x^2} = \frac{3x^2 + x - 1}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}
\]

Multiplying both sides by \(x^2(x+1)\), we get

\[
3x^2 + x - 1 = A(x)(x+1) + B(x+1) + Cx^2.
\]

Letting \(x=0\), this equation becomes

\[-1 = B\]

Letting \(x=-1\), this equation becomes

\[3(-1)^2 - 1 - 1 = C(-1)^2\]

\[1 = C\]

To find \(A\), we multiply out the right hand side & compare the terms to those on the left hand side.

\[3x^2 + x - 1 = Ax^2 + Ax + Bx + B + Cx^2\]

\[= Ax^2 + Cx^2 + Ax + Bx + B = (A+C)x^2 + (A+B)x + B\]

So, for this equality to hold, we must have

\[3 = A + C = A + 1\] (we must also have \(A + B = 1\), but this gives the same value for \(A\)).

So, \(\int \frac{x^3 + 4x^2 + x - 1}{x^3 + x^2} \, dx = \int 1 + \frac{2}{x} + \frac{4}{x^2} + \frac{1}{x+1} \, dx\]

\[= x + 2\ln|\mathbb{x}| + \frac{1}{x + \ln|x + 1|} + C.\]
4. \( \int \frac{x^2 - x + 6}{x^3 + 3x} \, dx \)

Sln: We proceed directly to partial fraction decomposition, as the rational function here is “proper”.

\[
\frac{x^2 - x + 6}{x^3 + 3x} = \frac{x^2 - x + 6}{x(x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}
\]

Multiplying both sides by \( x(x^2 + 3) \),

\[
x^2 - x + 6 = A(x^2 + 3) + (Bx + C)x.
\]

If \( x = 0 \), this equation becomes

\[
6 = 3A \quad \text{so} \quad A = 2.
\]

To find \( B \) & \( C \), we need to multiply out the RHS & compare it to the LHS:

\[
x^2 - x + 6 = 2(x^2 + 3) + (Bx + C)x
\]

\[
= 2x^2 + 6 + Bx^2 + Cx = (2 + B)x^2 + Cx + 6.
\]

So \( 2 + B = 1 \) and \( C = -4 \).

\[
B = -1
\]

\[
\int \frac{x^2 - x + 6}{x^3 + 3x} \, dx = \int \frac{2}{x} \, dx + \int \frac{-x - 1}{x^2 + 3} \, dx
\]

\[
= 2 \ln |x| - \int \frac{x}{x^2 + 3} \, dx - \int \frac{1}{x^2 + 3} \, dx \quad \text{(2)}
\]

To solve (1), use u-sub: \( u = x^2 + 3 \), \( du = 2x \, dx \), \( \frac{du}{2} = x \, dx \).

\[
\text{So} \quad \int \frac{x \, dx}{x^2 + 3} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |x^2 + 3|
\]

To solve (2):

\[
\int \frac{1}{x^2 + 3} \, dx = \frac{1}{3} \int \frac{1}{x^2 + 1} \, dx = \frac{1}{3} \int \left( \frac{1}{t} \right)^{2/3} + 1 \, dt = \frac{1}{3} \int \frac{dt}{t^{2/3} + 1}
\]

\[
= \frac{\sqrt{3}}{3} \arctan \left( \frac{\sqrt{3}}{3} \right).
\]

So

\[
\int \frac{x^2 - x + 6}{x^3 + 3x} \, dx = 2 \ln |x| - \frac{1}{2} \ln |x^2 + 3| - \frac{1}{\sqrt{3}} \arctan \left( \frac{x}{\sqrt{3}} \right) + C
\]