MATH 1A
Problem Solutions 9/22

1. Find the equation of a function with holes (removable discontinuities) at $x=0$ and $x=\pm 1$, vertical asymptotes at $x=2$ & $x=4$ and horizontal asymptote $y=\frac{7}{9}$.

<u>Soln:</u> Holes mean fcn must include $\frac{x(x-1)(x+1)}{x(x-1)(x+1)}$; vertical asymptotes mean fcn must include $\frac{1}{(x-2)(x-4)}$ as well.

$f(x)=\frac{x(x-1)(x+1)}{x(x-1)(x+1)(x-2)(x-4)}$ doesn't have the required horizontal asym. To have a nonzero horiz. asym., degree of top & bottom must be equal, but degree of numerator of f is 3, degree of denominator is 5, so we need to multiply f by some quadratic, say x^2+4x+2. $f(x)\cdot(x^2+4x+2)$ has a horiz. asymptote at 1; to get a horiz. asymptote at $\frac{7}{9}$, we should multiply by $\frac{7}{9}$.

So $g(x)=\frac{7x(x-1)(x+1)(x^2+4x+2)}{9x(x-1)(x+1)(x-2)(x-4)}$ is such a fcn.

2. $f(x)=\frac{1}{\sqrt{1-x}}$. Find $f'(-1)$ by finding $f'(a)$ & plugging in -1 for a. (use $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$). Then find $f'(-1)$ using the alternate definition of a derivative: $f'(-1)=\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}$.

<u>Soln:</u> $f'(a)=\lim_{h\to 0}\frac{\frac{1}{\sqrt{1-(a+h)}}-\frac{1}{\sqrt{1-a}}}{h}$ Getting common denominator

$$=\lim_{h\to 0}\frac{\frac{\sqrt{1-a}-\sqrt{1-a-h}}{\sqrt{1-a-h}\cdot\sqrt{1-a}}}{\frac{h}{1}}$$

$$=\lim_{h\to 0}\frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{(1-a-h)(1-a)}}\cdot\frac{\sqrt{1-a}+\sqrt{1-a-h}}{\sqrt{1-a}+\sqrt{1-a-h}}$$

$$=\lim_{h\to 0}\frac{1-a-(1-a-h)}{h\sqrt{(1-a-h)(1-a)}\,(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\sqrt{(1-a-h)(1-a)}\,(\sqrt{1-a}+\sqrt{1-a-h})}=\frac{1}{2(1-a)^{3/2}}$$

So $f'(-1)=\frac{1}{2(1-(-1))^{3/2}}=\frac{1}{2(2)^{3/2}}=\frac{1}{2^{5/2}}$.

$$f'(-1) = \lim_{x \to -1} \frac{\frac{1}{\sqrt{1-x}} - \frac{1}{\sqrt{2}}}{x-(-1)}$$

$$= \lim_{x \to -1} \frac{\frac{\sqrt{2} - \sqrt{1-x}}{\sqrt{2}\sqrt{1-x}}}{\frac{x+1}{1}}$$

$$= \lim_{x \to -1} \frac{\sqrt{2} - \sqrt{1-x}}{(x+1)(\sqrt{2}\sqrt{1-x})} \cdot \frac{\sqrt{2} + \sqrt{1-x}}{\sqrt{2} + \sqrt{1-x}}$$

$$= \lim_{x \to -1} \frac{\overbrace{2 - 1 + x}^{=x+1}}{(x+1)(\sqrt{2(1-x)})(\sqrt{2} + \sqrt{1-x})} = \lim_{x \to -1} \frac{1}{\sqrt{2(1-x)}\,(\sqrt{2} + \sqrt{1-x})}$$

$$= \frac{1}{\sqrt{4}\,(\sqrt{2} + \sqrt{2})}$$

$$= \frac{1}{2 \cdot 2\sqrt{2}} = \frac{1}{2^{5/2}}.$$

3. $g(t) = t^2 + 1$. Find $g'(a)$.

<u>Soln:</u> $g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h \to 0} \frac{(a+h)^2 + 1 - (a^2+1)}{h}$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 + 1 - a^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2}{h} = \lim_{h \to 0} \frac{h(2a+h)}{h} = \lim_{h \to 0} 2a + h = 2a.$$

4. There are 2 tangent lines of $g(t) = t^2 + 1$ that pass through the point $(0,0)$. What are the slopes of these 2 lines?

<u>Soln:</u> A line passing through the origin has an equation that looks like $y = mx$. ~~This~~ We're dealing with tangent lines, so $m = g'(a)$ for some a. We also know $(a, g(a))$ is on the tangent line. We just need to figure out what a is.

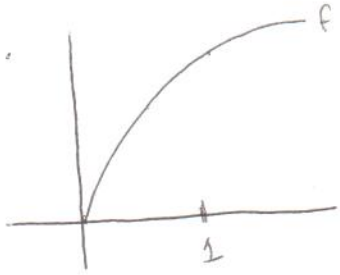
From problem 3, $g'(a) = 2a$. So the equation for the tangent lines in question looks like $y = 2ax$. Since the point $(a, g(a)) = (a, a^2+1)$ is on the line, we know $a^2 + 1 = 2a(a)$ (plugging (a, a^2+1) into eqn for tangent line)

We use this to solve for a: $a^2 + 1 = 2a^2$

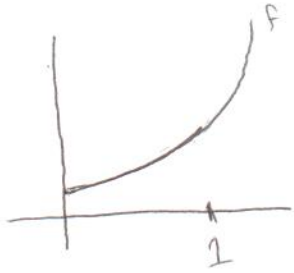
$1 = a^2$, so $a = \pm 1$.

So these tangent lines ~~are~~ hit g at -1 & 1, respectively. The slope of the first is $g'(-1) = -2$ & the slope of the 2nd is $g'(1) = 2$.

5.



A



B

For the fcns graphed in A & B, is $f'(1)$ larger or smaller than the slope of the secant line between $x=1$ & $x=1+h$ $(h>0)$? What about the slope of the secant line between $x=1-h$ & $x=1$ $(h>0)$?

<u>Soln:</u> (A) $f'(1)$ greater than slope of sec. line bet. $x=1$ & $x=1+h$

$f'(1)$ smaller ——————"—————— $x=1-h$ & $x=1$

(B) $f'(1)$ smaller than slope of sec. line bet. $x=1$ & $x=1+h$

$f'(1)$ greater ——————"—————— $x=1-h$ & $x=1$

Draw an example secant line & compare slope to slope of tangent line at 1 $(=f'(1))$