1. Find the equation of a function with holes (removable discontinuities) at \( x = 0 \) and \( x = \pm 1 \), vertical asymptotes at \( x = 2 \) and \( x = 4 \) and horizontal asymptote \( y = \frac{7}{4} \).

Solution: Holes mean the numerator must include \( \frac{x(x-1)(x+1)}{x(x-1)(x+1)} \); vertical asymptotes mean the denominator must include \( (x-2)(x-4) \) as well.

\[
f(x) = \frac{x(x-1)(x+1)}{x(x-1)(x+1)(x-2)(x-4)}
\]

doesn't have the required non-zero horizontal asymptote. To have a horizontal asymptote, degree of the top \( \frac{x(x-1)(x+1)}{} \) & bottom must be equal, but degree of the numerator of \( f \) is 3, degree of denominator is 5, so we need to multiply \( f \) by some quadratic, say \( x^2 + 4x + 2 \). \( f(x)(x^2 + 4x + 2) \) has a horizontal asymptote at 1, to get a horizontal asymptote at \( \frac{7}{4} \), we should multiply by \( \frac{7}{4} \).

So \( g(x) = \frac{7x(x-1)(x+1)(x^2 + 4x + 2)}{9x(x-1)(x+1)(x-2)(x-4)} \) is such a function.

2. \( f(x) = \frac{1}{\sqrt{1-x}} \). Find \( f''(-1) \) by finding \( f''(a) \) & plugging in \(-1\) for \( a \).

(Use \( f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \). Then find \( f''(-1) \) using the alternate definition of a derivative: \( f''(-1) = \lim_{x \to -1} \frac{f(x)-f(-1)}{x-(-1)} \).

Solution: \( f'(a) = \lim_{h \to 0} \frac{1}{\sqrt{1-(a+h)}} - \frac{1}{\sqrt{1-a}} \)

\[
= \lim_{h \to 0} \frac{\sqrt{1-a} - \sqrt{1-a-h}}{(1-a-h)(1-a-h)} \cdot \frac{1}{\sqrt{1-a}}
\]

\[
= \lim_{h \to 0} \frac{1-a-(1-a-h)}{(1-a-h)(1-a)} \cdot \frac{1}{\sqrt{1-a}}
\]

\[
= \lim_{h \to 0} \frac{h}{(1-a-h)(1-a)} \cdot \frac{1}{\sqrt{1-a}}
\]

\[
= \lim_{h \to 0} \frac{1}{(1-a-h)(1-a)} \cdot \frac{1}{\sqrt{1-a}} = \frac{1}{2(1-a)^{3/2}}
\]

So \( f''(-1) = \frac{1}{2(1-(-1))^{3/2}} = \frac{1}{2(2)^{3/2}} = \frac{1}{2^{7/2}} \).
\[ f'(1) = \lim_{x \to 1} \frac{1}{\sqrt{1-x}} - \frac{1}{\sqrt{2}} \]
\[ = \lim_{x \to 1} \frac{\sqrt{2} - \sqrt{1-x}}{\sqrt{2} \cdot \sqrt{1-x}} \cdot \frac{\sqrt{2} + \sqrt{1-x}}{\sqrt{2} + \sqrt{1-x}} \]
\[ = \lim_{x \to 1} \frac{2 - (1+x)}{(x+1)(\sqrt{2(1-x)})} \cdot \frac{\sqrt{2} + \sqrt{1-x}}{\sqrt{2} + \sqrt{1-x}} \]
\[ = \lim_{x \to 1} \frac{1}{\sqrt{2(1-x)}(\sqrt{2} + \sqrt{1-x})} \]
\[ = \frac{1}{\sqrt{4} (\sqrt{2} + \sqrt{2})} = \frac{1}{2 \cdot 2 \sqrt{2}} = \frac{1}{2 \sqrt{2}} . \]

3. \[ g(t) = t^3 + 1 \] . Find \( g'(a) \).

**Solution:**
\[ g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \]
\[ = \lim_{h \to 0} \frac{(a+h)^3 + 1 - (a^3 + 1)}{h} \]
\[ = \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 + 1 - a^3 - 1}{h} \]
\[ = \lim_{h \to 0} \frac{2ah + h^2}{h} = \lim_{h \to 0} \frac{h(2a + h)}{h} = \lim_{h \to 0} 2ah = 2a . \]

4. There are 2 tangent lines of \( g(t) = t^3 + 1 \) that pass through the point \((0,0)\). What are the slopes of these 2 lines?

**Solution:** A line passing through the origin has an equation that looks like \( y = mx \). We're dealing with tangent lines, so \( M = g'(a) \) for some \( a \). We also know \((a, g(a))\) is on the tangent line. We just need to figure out what \( a \) is.

From problem 3, \( g'(a) = 2a \). So the equation for the tangent lines in question looks like \( y = 2ax \). Since the point \((a, g(a)) = (a, a^3 + 1)\) is on the line, we know \( a^3 + 1 = 2a(a) \) (plugging \((a, a^3 + 1)\) into \(y = 2ax\) for tangent line).

We use this to solve \( a^3 + 1 = 2a^2 \)
\[ \Rightarrow a^3 - 2a^2 + 1 = 0 \]
\[ \Rightarrow a^2(a - 2) + 1 = 0 \]
\[ \Rightarrow (a^2 + 1)(a - 2) = 0 \]
\[ \Rightarrow a = \pm 1 \text{ or } a = 2 . \]

So these tangent lines hit \( g \) at \(-1, 1\) and \((1, 1)\), respectively. The slope of the first is \( g'(-1) = -2 \) & the slope of the 2nd is \( g'(1) = 2 \).
For the f(x) graphed in A & B, is \( f'(1) \) larger or smaller than the slope of the secant line between \( x=1 \) & \( x=1+h \) \((h>0)\)?

What about the slope of the secant line between \( x=1-h \) & \( x=1 \) \((h>0)\)?

**Soln:**

(A) \( f'(1) \) greater than slope of secant line bet. \( x=1 \) & \( x=1+h \)

\( f'(1) \) smaller \( x=1-h \) & \( x=1 \)

(B) \( f'(1) \) smaller than slope of secant line bet. \( x=1 \) & \( x=1+h \)

\( f'(1) \) greater \( x=1-h \) & \( x=1 \)

Draw an example secant line & compare slope to slope of tangent line at \( f(1) \).

\( f'(1) \)