MATH 1A Problem Solutions
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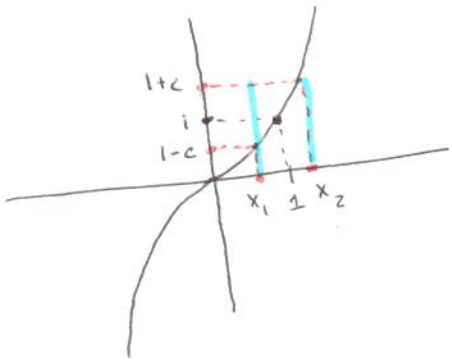
1. Use the graph of x^3 to find a δ such that $|x-1| < \delta \Rightarrow |x^3-1| < c$, where a) $c = \frac{1}{2}$ b) $c = \frac{1}{4}$.



Soln: a) $1+c = \frac{3}{2}$, $1-c = \frac{1}{2}$, so $x_1 = \sqrt[3]{\frac{1}{2}}$ & $x_2 = \sqrt[3]{\frac{3}{2}}$.

As long as x is between $\sqrt[3]{\frac{1}{2}}$ & $\sqrt[3]{\frac{3}{2}}$, $f(x)$ is within $\frac{1}{2}$ of $f(1) = 1$, but if x is any bigger, $f(x)$ isn't within $1/2$ of 1.

So $\delta = \min\{1 - \sqrt[3]{\frac{1}{2}}, \sqrt[3]{\frac{3}{2}} - 1\} = \sqrt[3]{\frac{3}{2}} - 1$ works

b) $1+c = \frac{5}{4}$, $1-c = \frac{3}{4}$, so $x_1 = \sqrt[3]{\frac{3}{4}}$, $x_2 = \sqrt[3]{\frac{5}{4}}$. Again, if x is between x_1 & x_2, $f(x)$ is within $\frac{1}{4}$ of $f(1)$, so $\delta = \min\{1 - \sqrt[3]{\frac{3}{4}}, \sqrt[3]{\frac{5}{4}} - 1\} = \sqrt[3]{\frac{5}{4}} - 1$ works

2. Show that $\lim_{x \to 0} 5x + 3 = 3$ using the δ-ε defn. of the limit.

Soln: Pick $\varepsilon > 0$. We want to find a δ such that if $\underset{|x| < \delta}{|x-0| < \delta}$,

Work backwards.

then $|5x + 3 - 3| < \varepsilon$

$|5x| < \varepsilon$.

$|5x| = 5|x|$, so $|5x| < \varepsilon$ only if $5|x| < \varepsilon$, $|x| < \frac{\varepsilon}{5}$. So $\delta = \frac{\varepsilon}{5}$ should work, so we try it.

If $|x| < \frac{\varepsilon}{5}$, then $|5x| = 5|x| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$, so ~~we get what we~~ $\delta = \frac{\varepsilon}{5}$ works.

3. Show that $\lim_{x \to 2} x^2 - 1 = 3$ using δ-ε defn.

Soln: Work backwards. We want to show that $|x^2 - 1 - 3| = |x^2 - 4|$ is small ~~trying~~ if we make $|x-2|$ small.

Pick $\varepsilon > 0$.

$$|x^2 - 4| = |(x-2)(x+2)| = \underbrace{|x-2|}_{\text{we're allowed to choose how small this is.}} \underbrace{|x+2|}_{\text{If x is close enough to 2, we can figure out how big this is.}}$$

Pick a ~~best~~ guess δ, say $\frac{1}{2}$. If $|x-2| < \frac{1}{2}$, then $-\frac{1}{2} < x - 2 < \frac{1}{2}$

$$\frac{3}{2} < x < \frac{5}{2}$$

So $\frac{3}{2} + 2 < x + 2 < \frac{5}{2} + 2$, which means $|x+2| < \frac{5}{2} + \frac{4}{2} = \frac{9}{2}$.

So for $\delta < \frac{1}{2}$, if $|x-2| < \delta$, then $|x+2| < \frac{9}{2}$.

So $|x-2||x+2| < |x-2| \cdot \frac{9}{2}$

If we want $|x-2| \cdot \frac{9}{2} < \varepsilon$, then $|x-2| < \frac{2}{9}\varepsilon$.

Choose $\delta = \min\{\frac{1}{2}, \frac{2}{9}\varepsilon\}$.

Then $|x-2| < \delta \Rightarrow |x^2-1-3| = |x^2-4|$

$$= \underbrace{|x-2|}_{<\frac{2}{9}\varepsilon}\underbrace{|x+2|}_{<\frac{9}{2}} < \frac{2}{9}\varepsilon \cdot \frac{9}{2} = \varepsilon.$$

4. Show that $\lim_{x \to 3^+} \frac{1}{x-3} = \infty$.

<u>Soln:</u> For this, we need to show that for any $M > 0$, we can find a $\delta > 0$ such that $3 < x < 3+\delta \Rightarrow \frac{1}{x-3} > M$.

~~Pick an~~ let $M > 0$, & work backwards. if $\frac{1}{x-3} > M$

then $1 > M(x-3)$

$\frac{1}{M} > x-3$, so we want $x-3 < \frac{1}{M}$, i.e. $\delta = \frac{1}{M}$.

Check that this δ works: if $3 < x < 3+\frac{1}{M}$, then $0 < x-3 < \frac{1}{M}$

and $M(x-3) < 1$

$M < \frac{1}{x-3}$.