1. Use the graph of $x^3$ to find a $\delta$ such that $|x-1|<\delta \implies |x^3-1|<\epsilon$.

where a) $\epsilon = \frac{1}{2}$ b) $\epsilon = \frac{1}{4}$

So $\delta$: a) $1+\epsilon = \frac{3}{2}$, $1-\epsilon = \frac{5}{4}$, so $x_1 = \frac{3}{2}$ and $x_2 = \frac{5}{4}$. $\implies$ $\implies$ $x_1 = \frac{3}{2}$, $x_2 = \frac{5}{4}$. Again, if $x$ is between $x_1$ and $x_2$, $f(x)$ is

As long as $x$ is between $\frac{3}{2}$ and $\frac{5}{4}$, $f(x)$ is within $\frac{1}{2}$ of $f(1) = 1$, but if $x$ is any bigger, $f(x)$ is not within $\frac{1}{2}$ of $1$.

So $\delta = \min \left(1, \frac{1}{2}, \frac{3}{2} - 1 \right) = \frac{3}{2} - 1 = \frac{1}{2}$ works.

b) $1+\epsilon = \frac{5}{4}$, $1-\epsilon = \frac{5}{4}$, so $x_1 = \frac{3}{2}$, $x_2 = \frac{5}{4}$. Again, if $x$ is between $x_1$ and $x_2$, $f(x)$ is

Within $\frac{1}{4}$ of $f(1)$, $\implies$ $\implies$ $\delta = \min \left(1, \frac{1}{2}, \frac{3}{2} - 1 \right) = \frac{3}{2} - 1 = \frac{1}{2}$ works.

2. Show that $\lim_{x \to 0} 5x+3 = 3$ using the $\epsilon$-$\delta$ defn. of the limit.

So $\delta$: Pick $\delta > 0$. We want to find a $\delta$ such that if $|x-0|<\delta$,

Then $|5x+3 - 3| < \epsilon$ if $|x| < \delta$.

$|5x + \frac{3}{5} - 3| < \epsilon$ if $|x| < \delta$.

So $\delta = \frac{\epsilon}{5}$ should work, so

If $|x| < \frac{\epsilon}{5}$, then $|5x + \frac{3}{5} - 3| < 5 \cdot \frac{\epsilon}{5} = \epsilon$.

3. Show that $\lim_{x \to 2} x^2 - 1 = 3$ using $\epsilon$-$\delta$ defn.

So $\delta$: Work backwards. We want to show that $|x^2 - 1 - 3| = |x^2 - 4| < \epsilon$ is

Small if we make $|x-2|$ small.

Pick $\delta > 0$. If $|x-2| < \delta$, then $|x^2 - 4| = |(x-2)(x+2)| = |x-2| |x+2|

We're allowed to choose how small this is.

Pick a $\delta$, say $\frac{1}{2}$. If $|x-2| < \frac{1}{2}$, then $\frac{1}{2} < x < 2$.

So $\frac{3}{2} < x < \frac{5}{2}$,

Which means $\delta = \frac{1}{2}$ works.
So \( |x-2|, \frac{a}{2} < \varepsilon \), then \( |x-2| < \frac{a}{2} \varepsilon \).

Choose \( \delta = \min \left( \frac{\varepsilon}{2}, \frac{a}{2} \varepsilon \right) \).

Then \( |x-2| < \delta \implies |x^2 - 1 - 3| = |x^2 - 4| \)

\[ = \left| \frac{|x-2|(x+2)}{2} \right| \leq \frac{\varepsilon}{2} \cdot \frac{a}{2} \varepsilon = \varepsilon. \]

4. Show that \( \lim_{x \to 3^+} \frac{1}{x-3} = \infty. \)

**Solu:** For this, we need to show that for any \( M > 0 \), we can find a \( \delta > 0 \) such that \( 3 < x < 3 + \delta \implies \frac{1}{x-3} > M. \)

Let \( M > 0 \), so work backwards, i.e. \( \frac{1}{x-3} > M \)

then \( 1 > M(x-3) \)

\( \frac{1}{m} > x-3 \), so we want \( x-3 \leq \frac{1}{m} \), i.e. \( \delta = \frac{1}{m} \).

Check that this \( \delta \) works: if \( 3 < x < 3 + \frac{1}{m} \), then \( 0 < x-3 < \frac{1}{m} \) and \( M(x-3) \leq 1 \).