Combinatorics of $\mathcal{X}$-variables in finite type cluster algebras

arXiv:1803.02492
Slides available at www.math.berkeley.edu/~msb

Melissa Sherman-Bennett, UC Berkeley

AMS Fall Central Sectional Meeting 2018
Cluster varieties in 2 flavors: $\mathcal{A}$-varieties (with $\mathcal{A}$-variables) and $\mathcal{X}$-varieties (with $\mathcal{X}$-variables)

$\mathcal{A}$-variables $\leftrightarrow$ cluster variables
$\mathcal{X}$-variables $\leftrightarrow$ coefficients

There is a duality between $\mathcal{A}$-varieties and $\mathcal{X}$-varieties (GHKK (2018)), but on the algebraic side much less is understood about $\mathcal{X}$-variables

$\mathcal{X}$-variables appear naturally in total positivity and in scattering amplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory (GGSVV (2014))
Let $\mathcal{F} \cong \mathbb{Q}(t_1, \ldots, t_n)$.

- An $\mathcal{X}$-seed $\Sigma$ in $\mathcal{F}$ is a pair $(\mathbf{x}, B)$, where $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_i \in \mathcal{F}$ and $B = (b_{ij})$ is a skew-symmetrizable $n \times n$ integer matrix.

- **Mutation** at $k \in \{1, \ldots, n\}$:

  \[
  (\mathbf{x}, B) \xrightarrow{\mu_k} (\mathbf{x}', B')
  \]

  where

  \[
  x_j' = \begin{cases} 
  x_j^{-1} & \text{if } j = k \\
  x_j(x_k + 1)^{-b_{kj}} & \text{if } b_{kj} \leq 0 \\
  x_j(x_k^{-1} + 1)^{-b_{kj}} & \text{if } b_{kj} > 0 
  \end{cases}
  \]

  and $B'$ is obtained from $B$ by matrix mutation at $k$. 

M. Sherman-Bennett (UC Berkeley)  
$\mathcal{X}$-variables in finite type  
AMS Fall Sectionals 2018
Definitions

Let $\mathcal{F} \cong \mathbb{Q}(t_1, \ldots, t_n)$.

- An $\mathcal{A}$-seed $\Sigma$ in $\mathcal{F}$ is a pair $(a, B)$, where $a = (a_1, \ldots, a_n)$ consists of algebraically independent elements of $\mathcal{F}$ and $B = (b_{ij})$ is a skew-symmetrizable $n \times n$ integer matrix.

- **Mutation** at $k \in \{1, \ldots, n\}$:

  $$(a, B) \xrightarrow{\mu_k} (a', B')$$

  where

  $$a'_j = \begin{cases} 
  a_k^{-1} \left( \prod_{b_{ik} > 0} a_i^{b_{ik}} + \prod_{b_{ik} < 0} a_i^{-b_{ik}} \right) & \text{if } j = k \\
  a_j & \text{if } j \neq k 
  \end{cases}$$

  and $B'$ is obtained from $B$ by matrix mutation at $k$. 
Seed Patterns

$\mathbb{T}_n$: $n$-regular tree with edges labeled with $1, \ldots, n$ so each vertex sees each label.

A seed pattern $S$ is a collection of seeds $\{\Sigma_t\}_{t \in \mathbb{T}_n}$ such that if $t \xrightarrow{k} t'$ in $\mathbb{T}_n$, then $\Sigma'_t = \mu_k(\Sigma_t)$.

An $\mathcal{A}$-seed pattern is finite type if it contains finitely many seeds.
Theorem (Fomin–Zelevinsky (2003))

An \( A \)-seed pattern \( S(a, B) \) is finite type if and only if \( B \) is mutation equivalent to a matrix whose Cartan companion is a finite type Cartan matrix.

Further, there is a bijection between \( A \)-variables and almost positive roots (positive or negative simple) in the corresponding root system.

Combinatorics of finite type \( A \)-seed patterns of classical type are encoded in tagged triangulations of certain marked surfaces (Fomin–Shapiro–Thurston (2008)).
Motivation

Theorem (Fomin–Zelevinsky (2003))

An $A$-seed pattern $S(a, B)$ is finite type if and only if $B$ is mutation equivalent to a matrix whose Cartan companion is a finite type Cartan matrix.

Further, there is a bijection between $A$-variables and almost positive roots (positive or negative simple) in the corresponding root system.

Combinatorics of finite type $A$-seed patterns of classical type are encoded in tagged triangulations of certain marked surfaces (Fomin–Shapiro–Thurston (2008)).

Question

Let $S(x, B)$ be an $X$-seed pattern of classical type. What can we say about its combinatorics?
Theorem (S.B. (2018))

Let $S$ be an $\mathcal{X}$-seed pattern of classical type and let $P$ be the corresponding marked surface. Then there is a bijection between the quadrilaterals (with a choice of diagonal) appearing in triangulations of $P$ and the $\mathcal{X}$-variables of $S$. 
The Answer

**Theorem (S.B. (2018))**

Let $S$ be an $\mathcal{X}$-seed pattern of classical type and let $P$ be the corresponding marked surface. Then there is a bijection between the quadrilaterals (with a choice of diagonal) appearing in triangulations of $P$ and the $\mathcal{X}$-variables of $S$.

**Classical types:**

<table>
<thead>
<tr>
<th>Type</th>
<th>$A_n$</th>
<th>$B_n, C_n$</th>
<th>$D_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\mathcal{X}(S)</td>
<td>$</td>
<td>$2\binom{n+3}{4}$</td>
</tr>
</tbody>
</table>

**Exceptional types:**

<table>
<thead>
<tr>
<th>Type</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\mathcal{X}(S)</td>
<td>$</td>
<td>770</td>
<td>2100</td>
<td>6240</td>
</tr>
</tbody>
</table>
$A$-seed patterns of classical types

$S$ a $A$-seed pattern of type $A_n (D_n)$, $P$ an $(n+3)$-gon (punctured $n$-gon).

- $\{A$-variables of $S\} \leftrightarrow \{\text{arcs of tagged triangulations of } P\}$
- $\{\text{seeds } \Sigma \text{ in } S\} \leftrightarrow \{\text{triangulations } T \text{ of } P\}$. If $\Sigma$ corresponds to $T$, the $A$-variables of $\Sigma$ correspond to the arcs of $T$ and the exchange matrix of $\Sigma$ can be obtained from $T$.
- Mutating $\Sigma$ at $k$ corresponds to flipping the $k^{th}$ arc in $T$.

There is an analogous story for types $B_n, C_n$ involving triangulations preserved by a particular group action.
A surjection...

Let $S$ be an $\mathcal{X}$-seed pattern of classical type, and $P$ be the appropriate marked surface.

- Think of seeds $(x, B)$ as triangulations $T$ of $P$ with arcs labeled by $\mathcal{X}$-variables ($B$ is the signed adjacency matrix of $T$).
- Mutating/flipping an arc $\gamma$ may change the labels of the arcs adjacent to $\gamma$.
- If we mutate away from the quadrilateral of an arc $\gamma$, the label of $\gamma$ does not change.

$$x'_j = \begin{cases} 
  x_j^{\frac{1}{b_{kj}}} & \text{if } j = k \\
  x_j(x_k + 1)^{-b_{kj}} & \text{if } b_{kj} \leq 0 \\
  x_j(x_k^{-1} + 1)^{-b_{kj}} & \text{if } b_{kj} > 0 
\end{cases}$$
Let $S$ be an $\mathcal{X}$-seed pattern of classical type, and $P$ be the appropriate marked surface.

- Think of seeds $(x, B)$ as triangulations $T$ of $P$ with arcs labeled by $\mathcal{X}$-variables ($B$ is the signed adjacency matrix of $T$).
- Mutating/flipping an arc $\gamma$ may change the labels of the arcs adjacent to $\gamma$.
- If we mutate away from the quadrilateral of an arc $\gamma$, the label of $\gamma$ does not change.
A surjection...

Fact (Fomin–Shapiro–Thurston (2008))

Let $T, T'$ be two tagged triangulations of a marked surface which both contain arcs $\tau_1, \ldots, \tau_s$. Then $T'$ can be obtained from $T$ by a sequence of arc flips avoiding $\tau_1, \ldots, \tau_s$.

So $\alpha : \{ q \cup \{ \gamma \} | q \text{ a quadrilateral with diagonal } \gamma \text{ in } P \} \to \mathcal{X}(S)$ is well-defined and surjective.
...which is injective.

**Proposition**

*The \( \mathcal{X} \)-variables associated to distinct quadrilaterals are distinct.*

Method of proof:

- Consider a particular \( \mathcal{A} \)-seed pattern \( \mathcal{R} \) of each type (can be found in e.g. *Intro. to cluster algebras*); \( \mathcal{A} \)-variables are rational functions on a vector space \( V \).
- Look at a related \( \mathcal{X} \)-seed pattern \( \mathcal{\hat{R}} \); \( \mathcal{X} \)-variables are rational functions of \( \mathcal{A} \)-variables in \( \mathcal{R} \).
- For any pair of \( \mathcal{X} \)-variables labeling diagonals of different quadrilaterals, verify that they are different functions on \( V \).
Corollaries and a Conjecture

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Type} & A_n & B_n, C_n & D_n \\
\hline
|\mathcal{X}| & 2\left(\frac{n+3}{4}\right) & \frac{1}{3}n(n+1)(n^2 + 2) & \frac{1}{3}n(n-1)(n^2 + 4n - 6) \\
|\mathcal{X}_{pc}| & n(n+1) & 2n^2 & 2n(n-1) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Type} & E_6 & E_7 & E_8 & F_4 & G_2 \\
\hline
|\mathcal{X}| & 770 & 2100 & 6240 & 196 & 16 \\
|\mathcal{X}_{pc}| & 72 & 126 & 240 & 48 & 12 \\
\hline
\end{array}
\]

Note: \(|\mathcal{X}_{pc}|\) is the number of \(\mathcal{X}\)-variables when we replace + with “tropical plus” in the \(\mathcal{X}\)-variable mutation formulas. The values follow from results of (Speyer–Thomas (2013)).
Corollaries and a Conjecture

Corollary

Let $S$ be an $\mathcal{X}$-seed pattern of type $Z_n$.

- The $\mathcal{X}$-variables in $S$ are in bijection with ordered pairs of exchangeable $\mathcal{A}$-variables in an $\mathcal{A}$-seed pattern of type $Z_n$.
- The $\mathcal{X}$-variables in $S$ are in bijection with ordered pairs of almost-positive roots with compatibility degree 1 in the root system of type $Z_n$.

Conjecture

Let $T$ be a tagged triangulation of a marked surface $(S, M)$, $B$ the signed adjacency matrix of $T$, and $S := S(x, B)$. Then the following map is a bijection:

$$\alpha : \{q \cup \{\gamma\} | q \text{ a quadrilateral with diagonal } \gamma \text{ in } (S, M)\} \rightarrow \mathcal{X}(S).$$
References


