Combinatorics of $\mathcal{X}$-variables in finite type cluster algebras

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Slides available at `www.math.berkeley.edu/~msb`

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Big Picture

- (Fock–Goncharov (2009)) Cluster varieties in 2 flavors: $\mathcal{A}$-varieties (with $\mathcal{A}$-variables) and $\mathcal{X}$-varieties (with $\mathcal{X}$-variables)

$$\begin{aligned} \mathcal{A}\text{-variables} &\leftrightarrow \text{cluster variables} \\ \mathcal{X}\text{-variables} &\leftrightarrow \text{coefficients} \end{aligned}$$

- There is a duality between $\mathcal{A}$-varieties and $\mathcal{X}$-varieties (GHKK (2018)), but on the algebraic side much less is understood about $\mathcal{X}$-variables
- $\mathcal{X}$-variables appear naturally in total positivity and in scattering amplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory (GGSVV (2014))

Definitions

Let $\mathcal{F} \cong \mathbb{Q}(t_1, \dots, t_n)$.

- An $\mathcal{X}$-**seed** Σ in $\mathcal{F}$ is a pair $(\mathbf{x}, B)$, where $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in \mathcal{F}$ and $B = (b_{ij})$ is a skew-symmetrizable $n \times n$ integer matrix.
- **Mutation** at $k \in \{1, \dots, n\}$:

$$(\mathbf{x}, B) \xrightarrow{\mu_k} (\mathbf{x'}, B')$$

where

$$x'_j = \begin{cases} x_j^{-1} & \text{if } j = k \\ x_j(x_k + 1)^{-b_{kj}} & \text{if } b_{kj} \leq 0 \\ x_j(x_k^{-1} + 1)^{-b_{kj}} & \text{if } b_{kj} > 0 \end{cases}$$

and B' is obtained from B by matrix mutation at k.

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Let $\mathcal{F} \cong \mathbb{Q}(t_1, \ldots, t_n)$.

- An $\mathcal{A}$-**seed** Σ in $\mathcal{F}$ is a pair $(\mathbf{a}, B)$, where $\mathbf{a} = (a_1, \ldots, a_n)$ consists of algebraically independent elements of $\mathcal{F}$ and $B = (b_{ij})$ is a skew-symmetrizable $n \times n$ integer matrix.
- **Mutation** at $k \in \{1, \ldots, n\}$:

$$(\mathbf{a}, B) \xrightarrow{\mu_k} (\mathbf{a'}, B')$$

where

$$a'_j = \begin{cases} a_k^{-1} \left(\prod_{b_{ik}>0} a_i^{b_{ik}} + \prod_{b_{ik}<0} a_i^{-b_{ik}} \right) & \text{if } j = k \\ a_j & \text{if } j \neq k \end{cases}$$

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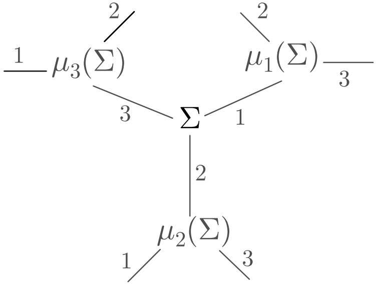
Seed Patterns

$\mathbb{T}_n$: n-regular tree with edges labeled with $1, \ldots, n$ so each vertex sees each label.

A **seed pattern** $\mathcal{S}$ is a collection of seeds $\{\Sigma_t\}_{t \in \mathbb{T}_n}$ such that if $t \overset{k}{\text{——}} t'$ in $\mathbb{T}_n$, then $\Sigma'_t = \mu_k(\Sigma_t)$.



An $\mathcal{A}$-seed pattern is **finite type** if it contains finitely many seeds.

Motivation

Theorem (Fomin–Zelevinsky (2003))

An $\mathcal{A}$-seed pattern $\mathcal{S}(\mathbf{a}, B)$ is finite type if and only if B is mutation equivalent to a matrix whose Cartan companion is a finite type Cartan matrix.
Further, there is a bijection between $\mathcal{A}$-variables and almost positive roots (positive or negative simple) in the corresponding root system.

Combinatorics of finite type $\mathcal{A}$-seed patterns of classical type are encoded in tagged triangulations of certain marked surfaces (Fomin–Shapiro–Thurston (2008)).

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Combinatorics of finite type $\mathcal{A}$-seed patterns of classical type are encoded in tagged triangulations of certain marked surfaces (Fomin–Shapiro–Thurston (2008)).

Question

Let $\mathcal{S}(\mathbf{x}, B)$ be an $\mathcal{X}$-seed pattern of classical type. What can we say about its combinatorics?

The Answer

Theorem (S.B. (2018))

Let $\mathcal{S}$ be an $\mathcal{X}$-seed pattern of classical type and let P be the corresponding marked surface. Then there is a bijection between the quadrilaterals (with a choice of diagonal) appearing in triangulations of P and the $\mathcal{X}$-variables of $\mathcal{S}$.

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Classical types:

Type	A_n	B_n, C_n	D_n
$\lvert\mathcal{X}(\mathcal{S})\rvert$	$2\binom{n+3}{4}$	$\frac{1}{3}n(n+1)(n^2+2)$	$\frac{1}{3}n(n-1)(n^2+4n-6)$

Exceptional types:

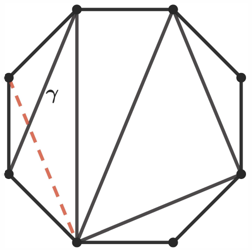
Type	E_6	E_7	E_8	F_4	G_2
$\lvert\mathcal{X}(\mathcal{S})\rvert$	770	2100	6240	196	16

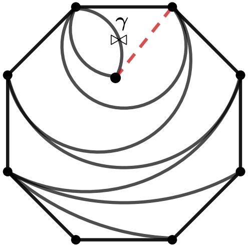
$\mathcal{A}$-seed patterns of classical types

$\mathcal{S}$ a $\mathcal{A}$-seed pattern of type A_n (D_n), P an $(n+3)$-gon (punctured n-gon).

- $\{\mathcal{A}$-variables of $\mathcal{S}\} \leftrightarrow$ $\{$arcs of tagged triangulations of $P\}$
- $\{$seeds Σ in $\mathcal{S}\} \leftrightarrow$ $\{$triangulations T of $P\}$. If Σ corresponds to T, the $\mathcal{A}$-variables of Σ correspond to the arcs of T and the exchange matrix of Σ can be obtained from T.
- Mutating Σ at k corresponds to flipping the k^{th} arc in T.

There is an analogous story for types B_n, C_n involving triangulations preserved by a particular group action.





A surjection...

Let $\mathcal{S}$ be an $\mathcal{X}$-seed pattern of classical type, and P be the appropriate marked surface.

- Think of seeds $(\mathbf{x}, B)$ as triangulations T of P with arcs labeled by $\mathcal{X}$-variables (B is the signed adjacency matrix of T).
- Mutating/flipping an arc γ may change the labels of the arcs adjacent to γ.
- If we mutate away from the **quadrilateral** of an arc γ, the label of γ does not change.

$$x_j' = \begin{cases} x_j^{-1} & \text{if } j = k \\ x_j(x_k + 1)^{-b_{kj}} & \text{if } b_{kj} \leq 0 \\ x_j(x_k^{-1} + 1)^{-b_{kj}} & \text{if } b_{kj} > 0 \end{cases}$$

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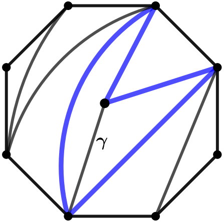

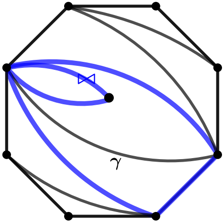

A surjection...

Fact (Fomin–Shapiro–Thurston (2008))

Let T, T' be two tagged triangulations of a marked surface which both contain arcs $\tau_1, \ldots, \tau_s$. Then T' can be obtained from T by a sequence of arc flips avoiding $\tau_1, \ldots, \tau_s$.

So $\alpha : \{q \cup \{\gamma\} | q$ a quadrilateral with diagonal γ in $P\} \to \mathcal{X}(\mathcal{S})$ is well-defined and surjective.

...which is injective.

Proposition

The $\mathcal{X}$-variables associated to distinct quadrilaterals are distinct.

Method of proof:

- Consider a particular $\mathcal{A}$-seed pattern $\mathcal{R}$ of each type (can be found in e.g. *Intro. to cluster algebras*); $\mathcal{A}$-variables are rational functions on a vector space V.
- Look at a related $\mathcal{X}$-seed pattern $\hat{\mathcal{R}}$; $\mathcal{X}$-variables are rational functions of $\mathcal{A}$-variables in $\mathcal{R}$.
- For any pair of $\mathcal{X}$-variables labeling diagonals of different quadrilaterals, verify that they are different functions on V.

Corollaries and a Conjecture

Type	A_n	B_n, C_n	D_n
$\lvert\mathcal{X}\rvert$	$2\binom{n+3}{4}$	$\frac{1}{3}n(n+1)(n^2+2)$	$\frac{1}{3}n(n-1)(n^2+4n-6)$
$\lvert\mathcal{X}_{pc}\rvert$	$n(n+1)$	$2n^2$	$2n(n-1)$

Type	E_6	E_7	E_8	F_4	G_2
$\lvert\mathcal{X}\rvert$	770	2100	6240	196	16
$\lvert\mathcal{X}_{pc}\rvert$	72	126	240	48	12

Note: $|\mathcal{X}_{pc}|$ is the number of $\mathcal{X}$-variables when we replace $+$ with "tropical plus" in the $\mathcal{X}$-variable mutation formulas. The values follow from results of (Speyer–Thomas (2013)).

Corollaries and a Conjecture

Corollary

Let $\mathcal{S}$ be an $\mathcal{X}$-seed pattern of type Z_n.

- *The $\mathcal{X}$-variables in $\mathcal{S}$ are in bijection with ordered pairs of exchangeable $\mathcal{A}$-variables in an $\mathcal{A}$-seed pattern of type Z_n.*
- *The $\mathcal{X}$-variables in $\mathcal{S}$ are in bijection with ordered pairs of almost-positive roots with* **compatibility degree** *1 in the root system of type Z_n.*

Conjecture

Let T be a tagged triangulation of a marked surface (S, M), B the signed adjacency matrix of T, and $\mathcal{S} := \mathcal{S}(\mathbf{x}, B)$. Then the following map is a bijection:
$\alpha : \{q \cup \{\gamma\} | q \text{ a quadrilateral with diagonal } \gamma \text{ in } (S, M)\} \to \mathcal{X}(\mathcal{S})$.

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