Combinatorics of cluster structures in Schubert varieties

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The set-up

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- The Schubert cell
  $\Omega_I := \{ V \in Gr_{k,n} : P_I(V) \neq 0, \ P_J(V) = 0 \text{ for } J < I \}$
- The open Schubert variety $X_I^\circ := \Omega_I \setminus \{ V \in \Omega_I : P_I P_{I_2} \cdots P_{I_n} = 0 \}$
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- Cluster algebra convention: Given a seed $(x, Q)$ with $x_{r+1}, \ldots, x_N$ frozen, $\mathcal{A}(x, Q)$ is the $\mathbb{C}[x_{r+1}^{\pm 1}, \ldots, x_N^{\pm 1}]$-algebra generated by the cluster variables.
Theorem (Scott ’06)

\( \mathbb{C}[\hat{\text{Gr}}_{k,n}] \) is a cluster algebra with seeds (consisting entirely of Plücker coordinates) given by Postnikov’s plabic graphs for \( \text{Gr}_{k,n} \).

(\( \hat{\text{Gr}}_{k,n} \) is the affine cone over \( \text{Gr}_{k,n} \) wrt Plücker embedding.)
Motivation

**Theorem (Scott ’06)**

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**Conjecture (Muller–Speyer ’16)**

Scott’s result holds if you replace \( \text{Gr}_{k,n} \) with an open positroid variety \( \pi_k(\mathcal{R}_{v,w}) \).
Main result

Theorem (SSW ’19)

\[ \mathbb{C}[\hat{X}_i^\circ] \text{ is a cluster algebra, with seeds (consisting entirely of Plücker coordinates) given by plabic graphs for } X_i^\circ. \]

(\(\hat{X}_i^\circ\) is the affine cone over \(X_i^\circ\) wrt Plücker embedding.)
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- More general result for open “skew Schubert” varieties \(\pi_k(R_{v,xv})\), where seeds for the cluster structure are given by *generalized* plabic graphs.
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• More general result for open “skew Schubert” varieties $\pi_k(\mathcal{R}_{\nu,x\nu})$, where seeds for the cluster structure are given by generalized plabic graphs.

• We use a result of (Leclerc ’16), who shows that coordinate rings of certain open Richardson varieties in the complete flag variety are cluster algebras.
Postnikov’s plabic graphs

A plabic graph of type \((k, n)\) is a planar graph embedded in a disk with

- \(n\) boundary vertices labeled \(1, \ldots, n\) clockwise.
- Internal vertices colored white and black.
- Boundary vertices are adjacent to a unique internal vertex.

![Diagram of a plabic graph](image)
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For generalized plabic graphs, we drop the condition that boundary vertices are labeled 1, \ldots, \(n\) clockwise.
Let $G$ be a reduced plabic graph of type $(k, n)$. The dual quiver $Q(G)$ is obtained by

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A trip in $G$ is a walk from boundary vertex to boundary vertex that
- turns maximally left at white vertices
- turns maximally right at black vertices
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Variables from plabic graphs

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**Aside:** The trip permutation of this graph is

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 4 & 5 & 1 & 3
\end{array}
\]
Face labels

If the trip $T$ ends at $j$, put a $j$ in all faces of $G$ to the left of $T$. Do this for all trips.

![Diagram with labeled faces]
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**Fact:** (Postnikov ’06) All faces of $G$ will be labeled by subsets of the same size (which is $k$).

To get cluster variables, we interpret each face label as a Plücker coordinate.
• Each reduced plabic graph corresponds to a unique positroid variety, determined by its trip permutation.
• The plabic graphs for $X_I^\circ$ have trip permutation

$$\pi_I = j_1j_2 \cdots j_{n-k}i_1i_2\cdots i_k$$

where $I = \{i_1 < i_2 < \cdots < i_k\}$ and $\{1, \ldots, n\} \setminus I = \{j_1 < j_2 < \cdots < j_{n-k}\}$. 
The trip permutation of $G$ is 24513, so this is a seed for $X_{\{1,3\}}$. 

So in the end...
Theorem

Let \( G \) be a reduced plabic graph corresponding to \( \chi_I \), and let \((x, Q(G))\) be the associated seed. Then \( A(x, Q(G)) = \mathbb{C}[\chi_I] \).

Corollaries:

• Classification of when \( A(x, Q(G)) \) is finite type
Applications

**Theorem**

Let $G$ be a reduced plabic graph corresponding to $X_i^\circ$, and let $(x, Q(G))$ be the associated seed. Then $A(x, Q(G)) = \mathbb{C}[X_i^\circ]$.

**Corollaries:**

- Classification of when $A(x, Q(G))$ is finite type
- From (Muller ’13) and (Muller-Speyer ’16), $A(x, Q(G))$ is locally acyclic, so it’s locally a complete intersection and equal to its upper cluster algebra
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Corollaries:

- Classification of when $A(x, Q(G))$ is finite type
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- From (Ford-Serhiyenko ’18), $A(x, Q(G))$ has green-to-red sequence, so satisfies the EGM property of (GHKK ’18) and has a canonical basis of $\theta$-functions parameterized by $g$-vectors.
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• How do seeds/cluster structure from generalized plabic graphs compare to seeds/cluster structure from normal plabic graphs?
• Is there a combinatorial characterization of compatibility of Plücker's for Schubert, skew Schubert? (In the Schubert case, there are seeds in \( \mathcal{A}(\mathfrak{x}, Q(G)) \) that consist entirely of Plücker coordinates, but the coordinates are not weakly separated.)