Metric properties of the affine building of SL_n

Slides available at `www.math.berkeley.edu/~msb`

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Buildings, Varieties, and Applications
MPI Leipzig

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- The **standard apartment** is

$$\mathcal{D} = \left\{ \begin{bmatrix} t^{a_1} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & t^{a_n} \end{bmatrix} \right\}$$

Coweight distance

Let $X, Y \in \mathcal{B}_n$. Using an automorphism of $\mathcal{B}_n$, translate X to I and Y to $Y' \in \mathcal{D}$, where Y' has a representative

$$y' = \begin{bmatrix} t^{-a_1} & 0 & \cdots & 0 \\ 0 & t^{-a_2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & t^{-a_n} \end{bmatrix}$$

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Then $d(X, Y) := (a_1, a_2, \ldots, a_n) - \frac{a_1+a_2+\cdots+a_n}{n}(1, 1, \ldots, 1)$.

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Basic properties: $d(X, Y) = -w_0 \cdot d(Y, X)$, d is translation-invariant, the triangle inequality holds but only triangles of certain edge lengths exist (Kapovich–Leeb–Millson). Used in Fontaine–Kamnitzer–Kuperberg.

Coweight geodesics

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- The **length** of a path $P = (X_0, X_1, \ldots, X_r)$ is

$$\mathrm{len}(P) = \sum_{i=1}^{r} d(X_{i-1}, X_i).$$

P is a **coweight geodesic** if $\mathrm{len}(P)$ is minimal among all paths $X_0 \rightsquigarrow X_r$.

Coweight geodesics

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Moreover, $M(X, Y) := \{Z \in \mathcal{B}_n : Z \text{ on a coweight geodesic } X \rightsquigarrow Y\}$ is finite and consists of the lattice points in a zonotope (Minkowski sum of edges).

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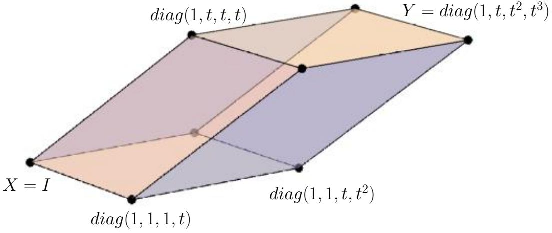

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- Suppose $d(X, Y) = (v_1, \dots, v_n)$. For $j = 1, \dots, n-1$, the **j-distance** from X to Y is $d_j(X, Y) := v_1 + \cdots + v_j$.

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- If $\mathrm{len}(P) = (w_1, \dots, w_n)$, the **j-length** of P is $\mathrm{len}_j(P) := w_1 + \cdots + w_j$.
- A path $P = (X, \dots, Y)$ is **j-minimal** if $\mathrm{len}_j(P)$ is minimal among all paths $X \rightsquigarrow Y$

Basic properties of j-minimal paths

Call $M_j(X,Y) := \{Q : Q \text{ on a } j\text{-minimal path } X \rightsquigarrow Y\}$ the *j-minimizing set* for X and Y. For an apartment $\mathcal{A}$, let $M_j^{\mathcal{A}}(X,Y) := M_j(X,Y) \cap \mathcal{A}$.

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- $\cap_{j=1}^{n-1} M_j(X, Y) = M(X, Y)$
- $M_j(X, Y) = \{Q : Q \text{ minimizes } d_j(X, Q) + d_j(Z, Q)\}$

j-minimal paths between adjacent points

Let $\Delta_i := \text{diag}(1, \ldots, 1, \underbrace{t, \ldots, t}_{i})$.

Proposition (SB–Zhang, '19+)

Let $j \in \{1, \ldots, n-1\}$. Then

$$M_j^{\mathcal{D}}(I, \Delta_i) = \begin{cases} \{[\text{diag}(1, \ldots, 1, t^{a_1}, \ldots, t^{a_i})] : a_k \in \{0, 1\}\} & j < n - i \\ \{[\text{diag}(t^{b_1}, \ldots, t^{b_{n-i}}, t, \ldots, t)] : b_k \in \{0, 1\}\} & j > n - i \\ M_{j-1}^{\mathcal{D}}(I, \Delta_i) \cup M_{j+1}^{\mathcal{D}}(I, \Delta_i) & j = n - i \end{cases}$$

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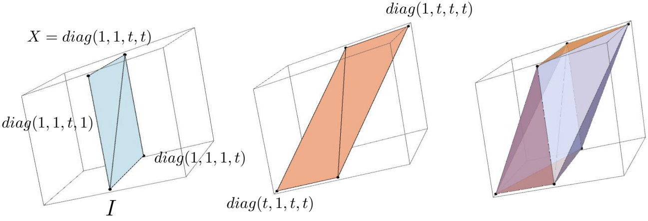

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Corollary

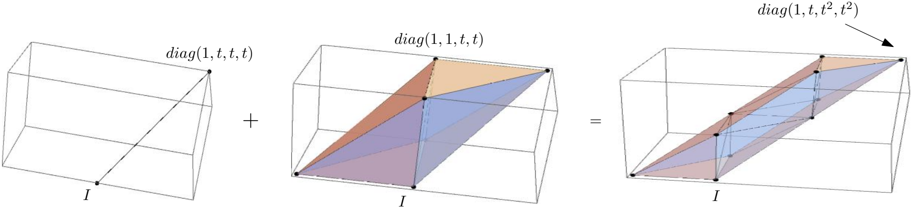
Description of $M_j^{\mathcal{A}}(X, Y)$ for X, Y adjacent and $\mathcal{A}$ an apartment containing X, Y.

j-minimal paths between arbitrary points

If we restrict our attention to a single apartment:

Theorem (SB–Z, '19+)

Suppose $X = \sum_{k=1}^{r} \Delta_{i_k}$. *Then* $M_j^{\mathcal{D}}(I, X) = \sum_{k=1}^{r} M_j^{\mathcal{D}}(I, \Delta_{i_k})$.



$M_2(I, \Delta_3)$ $M_2(I, \Delta_2)$ $M_2(I, \Delta_3 + \Delta_2)$

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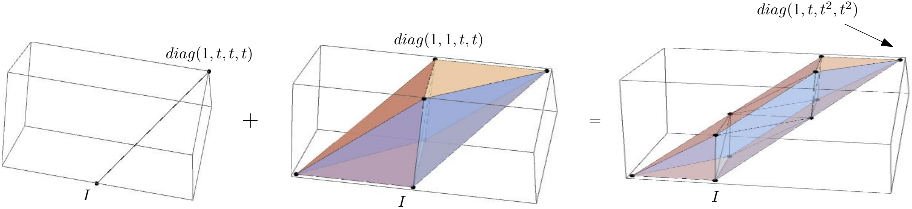

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Corollary

Description of $M_j^{\mathcal{A}}(X, Y)$ *for* X, Y *arbitrary and* $\mathcal{A}$ *an apartment containing* X, Y.

j-minimal paths between arbitrary points

If we look at all of $\mathcal{B}_n$:

Theorem (SB–Z, '19+)

Let P be a coweight geodesic from X to Y. Then all j-minimal paths from X to Y can be obtained from P by repeatedly applying the following 3 moves:

- $(+)$ *If $Z \in M_j(A, B)$, replace an edge $A \to B$ with the 2-path $A \to Z \to B$.*
- $(-)$ *If A, B are adjacent and $Z \in M_j(A, B)$, replace $A \to Z \to B$ with $A \to B$.*
- $(\diamond)$ *If A, B are not adjacent and $M_j^{\mathcal{A}}(A, B) = \{A, B, U, Z\}$ for $\mathcal{A} \ni A, B$, replace $A \to U \to B$ with $A \to Z \to B$.*

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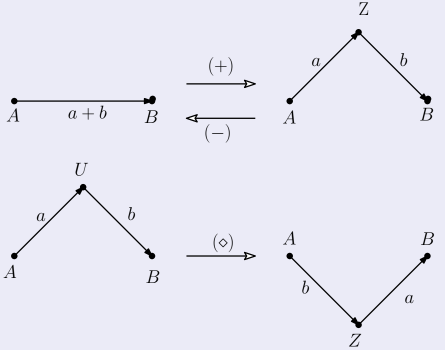

Motivation

For integers i, j, k with $i + j + k = n$ and $X, Y, Z \in \mathcal{B}_n$ let

$$g_{ijk}^{XYZ}(Q) := d_i(Q, X) + d_j(Q, Y) + d_k(Q, Z).$$

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$$\begin{aligned} M_j(X, Y) &= \{Q : Q \text{ minimizes } d_j(X, Q) + d_j(Q, Y)\} \\ &= \{Q : Q \text{ minimizes } d_{n-j}(Q, X) + d_j(Q, Y)\} \end{aligned}$$

is the set of minimizers of $g_{(n-j),j,0}^{XYZ}$.

Conjectures

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Suppose X, Y, Z *form a 3-simplex, and let* $i, j, k > 0$ *be integers with* $i + j + k = n$*. Then there exist integers* a, b, c *such that* $M_a(X, Y) \cap M_b(Y, Z) \cap M_c(X, Z)$ *is the set of points minimizing* g_{ijk}^{XYZ}*.*

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Question

To what extent is the minimizing set of g_{ijk}^{XYZ} for arbitrary X, Y, Z controlled by the minimizing sets of g_{ijk}^{ABC}, where A, B, C form a 3-simplex?

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For i, j, k *integers with* $i + j + k = n$ *and* $X, Y, Z \in \mathcal{B}_n$*,*
$g_{ijk}^{XYZ}(Q) = d_i(Q, X) + d_j(Q, Y) + d_k(Q, Z)$ *is L-convex, in the sense of (Hirai, '17).*

Thanks for listening!