Kazhdan-Lusztig immanants & k-positive matrices

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(joint w/ S. Chepuri; arXiv: 2002.07851)

Defn: $f: S_n \to \mathbb{C}$. The immanant associated to f is $\mathrm{Imm}_f : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{C}$
$$M=(m_{ij}) \longmapsto \sum_{w\in S_n} f(w) \underbrace{m_{1,w(1)} \cdots m_{n,w(n)}}_{\substack{\shortparallel \\ \underline{m}_w}}$$

Ex: • $f(w) = (-1)^{\ell(w)}$. Then $\mathrm{Imm}_f(M) = \det(M)$
• χ the character of an irreducible S_n-representation. $\mathrm{Imm}_\chi(M)$ is a character immanant.

Defn: $M \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is totally $\begin{cases}\text{positive (TP)} \\ \text{nonnegative (TNN)}\end{cases}$ if all minors are $\begin{cases}\text{positive} \\ \text{non negative}\end{cases}$. M is k-positive if all minors of size at most k are positive.
(Note: k-positive matrices have real positive entries)

Stembridge '92, "Some conjectures on immanants": Conj: If M TNN, $\mathrm{Imm}_\chi(M) \geq 0$ for χ irred. character.

Question: If M k-pos, are certain monomial immanants pos?

Aside: Stembridge made other conjectures on "positivity" of $\mathrm{Imm}_\chi(M)$ assuming "positivity" of minors of M (e.g. Conj: Imm_χ (gen'lized Jacobi-Trudi mtx) is Schur-positive, later proved by Haiman)

Thm: [Stembridge '91] If M TNN, $\mathrm{Imm}_\chi(M) \geq 0$ for χ irred. character.
↳ relies on factorization result for TNN matrices, which doesn't have a k-positive analogue.

• Rhoades & Skandera '06 define K-L immanants (inspired by Haiman's use of K-L thy to solve conj) & use results of Stembridge, Haiman to show they have nice positivity properties.

Defn: $v \in S_n$. The Kazhdan-Lusztig (K-L) immanant assoc. to v is $\mathrm{Imm}_v(M) := \sum_{w\in S_n} (-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1)\, \underline{m}_w$,
where w_0 = longest permutation = $\begin{matrix}1 & 2 & \cdots & n \\ \downarrow & \downarrow & & \downarrow \\ n & n-1 & \cdots & 1\end{matrix}$ & $P_{x,y}(q)$ is K-L polynomial for S_n.

Rmks on K-L polys: $\leq$ is Bruhat order on S_n
1) $P_{x,y}(q) = 0$ if $x \not\leq y$ 2) $P_{x,y}(q) = 1$ if $x = y$ 3) $P_{x,y}(0) = 1$ if $x \leq y$
• Satisfy a recurrence involving R-polynomials & some degree conditions
• Related to nice basis of Hecke alg. for S_n
(*) $P_{x,y}(q)$ is Poincaré poly of local intersection cohomology of Schubert variety X_x at any pt in X_y. [K-L '80]

Note: 1) ⇒ $\mathrm{Imm}_v(M) = \sum_{w \geq v} (-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1)\, \underline{m}_w$

Ex: • $\mathrm{Imm}_{w_0}(M) = P_{e,e}(1)\, \underline{m}_{w_0} \overset{(2)}{=} m_{w_0}$

• $\mathrm{Imm}_e(M) = \sum_{w\in S_n} (-1)^{\ell(w)} P_{w_0 w, w_0}(1)\, \underline{m}_w$ $P_{x,w_0}(q) = 1\ \forall x \in S_n$, so
$= \det(M)$

Thm: [Rhoades-Skandera, '06] If M is TNN, then $\mathrm{Imm}_v(M) \geq 0\ \forall v \in S_n$. Moreover, $\mathrm{Imm}_\chi(M)$ is non-negative linear combination of K-L immanants.
↳ implies Stembridge's previous positivity result

Aside: Can write dual canonical basis of $\mathcal{O}(GL_n)$ in terms of K-L immanants [Skandera

Question: Fix $v \in S_n$. For which k is $\mathrm{Imm}_v(M) > 0$ for all M k-pos?
Notice for $v = e$, only $k = n$.

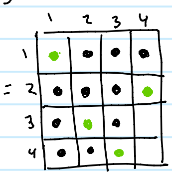
Defn: $v, w \in S_n$ & $v \leq w$. The graph of $[v,w]$ is $\Gamma[v,w] := \{(i, u_i) \mid v \leq u \leq w,\ i = 1, \ldots, n\}$.

Defn: $v, w \in S_n$ & $v \le w$. The graph of $[v,w]$ is $\Gamma[v,w] := \{(i, u_i) \mid v \le u \le w,\ i = 1, \dots, n\}$.

Thm: [Chepuri–SB '20] Suppose $v \in S_n$ avoids 1324 & 2143. If the largest square region of $\Gamma[v, w_0]$ has size k, then $\mathrm{Imm}_v(M) > 0$ for all M k-positive.

Pattern avoidance: v avoids
- 1324 if we never see $i \dots j \dots k \dots l$ with $i < k < j < l$
- 2143 — " — $i \dots j \dots k \dots l$ with $j < i < l < k$

Ex: $v = 1423$

$\Gamma[v, w_0] =$ 

lgst sq. has size 3, so $\mathrm{Imm}_{1423}(M) > 0$ for M 3-positive.

Ingredients in pf

- If y is 4231- & 3412-avoiding, then [Lakshmibai–Sandhya '90] & [K-L '80] $\Rightarrow P_{x,y}(q) = 1 \ \forall x \le y$

↳ if v is 1324 & 2143-avoiding (so $w_0 v$ is 4231- & 3412-avoiding) then
$$\mathrm{Imm}_v(M) = (-1)^{\ell(v)} \sum_{w \ge v} (-1)^{\ell(w)} \, m_w.$$

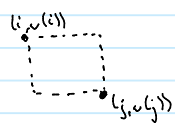
↳ $\mathrm{Imm}_v(M)$ is a det missing some terms. (up to sign)

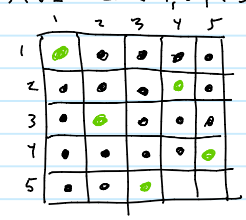
Q: Can we zero out entries of M so det. of resulting mtx is $\mathrm{Imm}_v(M)$?

Notation: $P \subseteq \{1, \dots, n\}^2$. The restriction of M to P is $M|_P = (\tilde{m}_{ij})$ where $\tilde{m}_{ij} := \begin{cases} m_{ij} & \text{if } (i,j) \in P \\ 0 & \text{else} \end{cases}$.

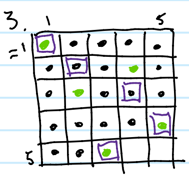
- So Q is: $\exists P$ s.t. $\det(M|_P) = \mathrm{Imm}_v(M)$?

↳ Because of terms in $\mathrm{Imm}_v(M)$, only possibility for P is $\Gamma[v, w_0] = \{(i, w(i)) \mid v \le w \ \& \ i = 1, \dots, n\}$.

How to draw $\Gamma[v, w_0]$:
1) Put a dot in $(i, v(i))$ $\forall i$
2) Fill in all rectangles like 

Ex: $v = 14253$ 

WARNING: $\det(M|_{\Gamma[v,w_0]}) = \sum_{\substack{w \\ \Gamma(w) \subseteq \Gamma[v,w_0]}} (-1)^{\ell(w)} \, m_w$ & in general it is NOT the case that $\{w \mid \Gamma(w) \subseteq \Gamma[v, w_0]\} = [v, w_0]$

e.g. $v = 14253$. $\Gamma[v, w_0] =$ 

Consider $x = 12453$. $x \not\geq v$ & $\Gamma(x) \subseteq \Gamma[v, w_0]$, so $\det(M|_{\Gamma[v,w_0]}) \ne \sum_{w \ge v} (-1)^{\ell(w)} \, m_w$.

Notice: The pattern 1324 was problematic.

Thm: [Sjöstrand '07] $\{w \mid \Gamma(w) \subseteq \Gamma[v, w_0]\} = [v, w_0] \iff v$ avoids 1324, 24153, 31524, & 426153.

Cor: If v avoids 1324 & 2143, then $\mathrm{Imm}_v(M) = (-1)^{\ell(v)} \det(M|_{\Gamma[v,w_0]})$.

- Now just need to determine sign of $\det(M|_{\Gamma[v,w_0]})$ when M is k-positive.

- Main tool is Desdanot–Jacobi formula

N an $n \times n$ mtx; N_I^J submatrix obtained by removing rows I & cols J from N. Choose rows $1 \le a < a' \le n$ & cols $1 \le b < b' \le n$. Then
$$\det(N) \det(N_{a,a'}^{b,b'}) = \det(N_a^b) \det(N_{a'}^{b'}) - \det(N_{a'}^{b}) \det(N_a^{b'}).$$

and

Prop: Sup. v is 1324 & 2143-avoiding. Then $\det\left[(M|_{\Gamma[v,w_0]})_i^{v_i} \right] = \det\left[(M_i^{v_i})|_{\Gamma[v', w_0]} \right]$ where v' obtained from v by deleting v_i from v in one-line notation & shifting other numbers appropriately.

(e.g. $v = 256134$; $i = 5$, $v_i = 3$; $v' = 24513$)

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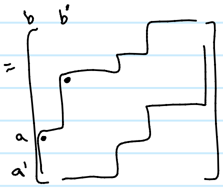
<u>Thm</u>: Sup. v is 1324 & 2143-avoiding. Sup. k is the size of the largest square in $\Gamma[v,w_0]$. Then $\mathrm{Imm}_v(M) > 0$ for all M k-positive. (equivalently, $(-1)^{\ell(v)} \det(M|_{\Gamma[v,w_0]}) > 0$ $\forall M$ k-pos)

<u>Pf sketch</u>: Induct on n. Can assume $\Gamma[v,w_0]$ is not block-antidiagonal. Choose a, a', b, b' like

$\Gamma[v,w_0] =$ 

& then analyze signs of terms in Desdanot-Jacobi, using Prop.

If $\Gamma[v,w_0]$ is a partition (or the complement of a partition), can show separately that $\det(M|_{\Gamma[v,w_0]})$ has sign $(-1)^{\ell(v)}$ (again using Desdanot-Jacobi)

<u>Question</u>: Connections to rep thry?

<u>Question</u>: Thm shows we can predict sign of $\det(M|_{\lambda/\mu})$ for some λ/μ (if M k-positive & size of largest sq in λ/μ is k).
(In fact, can use Desdanot-Jacobi to predict sign for $\mu = \emptyset$ or $\lambda = (n, \ldots, n)$).
However, this is not true of arbitrary λ/μ.
e.g. $\lambda = (4,4,3,2)$, $\mu = (2,1)$ $\begin{bmatrix} 0 & 0 & * & * \\ 0 & * & * & * \\ * & * & * & 0 \\ * & * & 0 & 0 \end{bmatrix} = M|_{\lambda/\mu}$. $\det(M|_{\lambda/\mu})$ takes both pos & neg. values on TP matrices.

For which λ/μ is the sign of $\det(M|_{\lambda/\mu})$ predictable? (w/ appropriate positivity constraints on M - either TP or k-pos).