# Coxeter Matroids

## An Introduction

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# Coxeter Combinatorics

**Coxeter combinatorics** is the field which studies extensions of common combinatorial objects to other Lie types.

Examples of objects in this field:

- Signed Graphs (Zaslavsky)
- Parsets (Reiner)
- Generalized Associahedra and Cambrian Fans (Fomin-Zelevinsky, Reading-Speyer)
- Signed Set Partitions
- Coxeter Generalized Permutahedra (Ardila-Castillo-Eur-Postnikov)

Coxeter combinatorics is a **dream** not a promise! It is possible that some objects do not have generalizations.

# Matroids

Matroids are among the hardest objects to generalize in this way due to their many definitions.

Some important definitions and where they are used:

- **Basis Exchange** - Key to the intuition of matroids capturing linear independence.
- **Flat Partition Axiom** - Used in the Chow ring of matroids and in tropical geometry.
- **Rank Function** - Captures the idea of dimension. Important everywhere
- **Greedy Algorithm** - Idea behind the thin stratification of $\mathrm{Gr}(k, n)$.
- **Circuit Axiom** - Connected to Whitney's original motivation of graph coloring.
- **Matroid Polytope** - Relates matroids to generalized permutahedra

## Fixing Notation

I will use the following notation:

- Let $\Phi$ be a root system (usually of type A, B, C, or D).
- Let $\Delta = [n]$ denote a simple system.
- Let $W$ be its Weyl group with presentation given by simple reflections $s_i$
- Let $\mathcal{A}$ be the Coxeter arrangement of $\Phi$. (Sometimes viewed as a fan).
- For any $I \subseteq [n]$, let $W_I$ be the parabolic subgroup of $W$ given by $I$.
- Let $\leq$ denote the Bruhat order on $W$ and $W/W_I$ (induced by $\Delta$).
- For any $w \in W$, let $\leq_w$ denote the shifted Bruhat order

$$v_1 W_I \leq_w v_2 W_I \iff w^{-1} v_1 W_I \leq w^{-1} v_2 W_I.$$

# Coxeter Greedy Algorithm

Our current definition of Coxeter matroids comes from Gelfand and Serganova's study of thin stratifications of $G/P$.

**Definition (Coxeter Greedy Algorithm)**

A **Coxeter Matroid** of type $(W, W_I)$ is a subset $M \subseteq W/W_I$ such that for any $w \in W$ there is a unique $\leq_w$-minimal element in $M$.

Let's compare this to our previous definitions.

# Relation to Ordinary Matroids

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We can identify $W/W_I$ with the permutations with single descent at position $k$ (in one-line notation).

$$S_4/W_{[4]-2} = \{12|34, 13|24, 14|23, 23|14, 24|13, 34|12\}.$$

We can further identify this with subsets of $[n]$ of cardinality $k$.

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**Exercise:** Show that the Bruhat order on $S_n/W_I$ corresponds to the Gale order on subsets of size $k$ of $[n]$.

A shift of the Bruhat order corresponds to a different choice of linear order on $[n]$.

# Relation to Flag Matroids

Let $W = S_5$ and $I = [5] - \{2, 4\}$. Then $W/W_I$ can be identified with permutations with a descent at 2 and 4.

$$S_5/W_{5-\{2,4\}} = \{12|34|5, 13|24|5, \ldots, 34|12|5, 35|14|2, \ldots\}.$$

We can further identify these with 2-step flags where the first flag has cardinality 2 and the second has cardinality 4.

$$S_5/W_{5-\{2,4\}} = \{12 \subseteq 1234, 13 \subseteq 1234, \ldots, 34 \subseteq 1234, 35 \subseteq 1345, \ldots\}.$$

**Annoying Exercise:** Show that the Bruhat order on $W/W_I$ corresponds to the Gale order of flags.

# Relation to Symplectic Matroids

Let $W$ be the type $C_5$ Weyl group. This corresponds to signed permutations which are maps $\phi : [\pm 5] \to [\pm 5]$ such that $\phi(\bar{i}) = \overline{\phi(i)}$. In one-line notation:

$$C_5 = \{31452, 3\bar{1}4\bar{5}2, 123\bar{4}5, \dots\}.$$

We will use the generators of $C_5$ given by $s_1, s_2, s_3, s_4, s_5$ where $s_i = (i, i+1)$ for $1 \leq i \leq 4$ and $s_5 = (5, \bar{5})$.

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Let $I = [5] - \{5\}$, then $W_I = S_5$ and $W/W_I \cong (\mathbb{Z}/2\mathbb{Z})^5$ can be identified with admissible subsets of $[\pm 5]$ of size 5.

$$W/W_I = \{12\bar{3}45, 123\overline{45}, \overline{123}45, \ldots\}.$$

Let $I = [5] - \{2\}$, then $W/W_I$ can be identified with admissible subsets of size 2

$$W/W_I = \{13, 2\bar{3}, 14, \overline{25}, \ldots\}.$$

Last time we discussed the Gale order on signed subsets. Consider the usual linear order

$$\overline{n} < \overline{n-1} < \cdots \overline{1} < 1 < \cdots < n.$$

A signed subset $\{i_1 < i_2 < \cdots < i_k\}$ is less than a signed subset $\{j_1 < \cdots < j_k\}$ in the **Gale order** iff $i_\ell < j_\ell$ for all $1 \leq \ell \leq k$.

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**Claim/Exercise:** The Bruhat order on $W/W_I$ of type $C_n$ agrees with the Gale order on admissible subsets.

The shifted Bruhat orders agree with different admissible linear orders on $[\pm n]$. Admissible here means that if $i < j$ then $\bar{j} < \bar{i}$.

Here are some examples coming from Bruhat intervals.

1. The trivial examples $W/W_I$ and any singleton $\{wW_I\}$.
2. The (shifted) **Coxeter Schubert matroid** $\Omega_{wW_I} = \{vW_I \mid wW_I \leq_u vW_I\}$ for any $u \in W$.
3. The interval Coxeter matroid $M = [w_1 W_I, w_2 W_I]$ with $w_1 \leq w_2$.
4. Let $\pi : W \to W/W_I$ be the natural projection map. Let $[w_1, w_2]$ be an interval in $W$. Then, $\pi([w_1, w_2])$ is a Coxeter matroid polytope.

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**Exercise:** Convince yourself that there are examples of (4) that are not examples of (3).

Beautifully, the ordinary matroids arising from example (4) are exactly the class of positroids (Tsukerman-Williams).

Now fix a crystallographic root system $\Phi$. Then, we have an associated lattice $\Lambda$ called the weight lattice of $\Phi$. This lattice is generated by certain lattice points $\rho_1, \ldots, \rho_n$ called the **fundamental weights** of $\Phi$. They are in bijection with the simple roots.

For any $I \subseteq [n]$ let $\rho_I = \sum_{i \in I} \rho_i$.

**Definition (Type $A_{n-1}$ Root Datum)**

- $\Phi = \{e_i - e_j\}_{i,j \in [n]}$
- $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$
- Fundamental weights: $\{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_n\}$.

# Coxeter Matroid Polytopes

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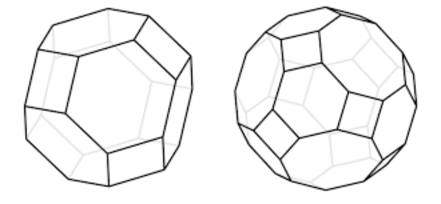
- Fundamental weights: $\{e_1, e_1 + e_2, \dots, e_1 + e_2 + \cdots + e_n\}$.

**Fact:** There is a natural action of $W$ on the weight lattice. Further the sum $\rho_I$ is stabilized by $W_{[n] \setminus I}$. Thus, the orbit of $\rho_I$ is in bijection with $W / W_{[n] \setminus I}$.

### Definition

A **Coxeter matroid polytope** of type $(W, W_I)$ is a polytope whose vertices are contained in the orbit of $\rho_{[n]\setminus I}$ under $W$ and whose edge directions are parallel to the roots in $\Phi$.



# Type $A_n$ Coxeter Matroid Polytopes

## Definition (Type $A_{n-1}$ Root Datum)

- $\Phi = \{e_i - e_j\}_{i \neq j \in [n]}$
- $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$
- Fundamental weights: $\{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_n\}$. (Slight lie)

For example, the fundamental weight $\rho_k = e_1 + \cdots + e_k$ is stabilized by $W_{[n]\setminus k} \cong S_k \times S_{n-k}$. The orbits correspond to the different 0-1 vectors with exactly $k$ ones. This agrees with the usual matroid polytopes.

# Type $B_n$ Coxeter Matroid Polytopes

## Definition (Type $B_n$ Root Datum)

- $W \cong$ {signed permutations of n}
- $\Phi = \{\pm e_i \pm e_j\}_{i \neq j \in [n]} \cup \{\pm e_i\}_{i \in [n]}$
- $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$
- Fundamental weights:
  $\{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_{n-1}, (e_1 + e_2 + \cdots + e_n)/2\}$.

# Type $B_n$ Coxeter Matroid Polytopes

## Definition (Type $B_n$ Root Datum)

- $W \cong \{\text{signed permutations of n}\}$
- $\Phi = \{\pm e_i \pm e_j\}_{i \neq j \in [n]} \cup \{\pm e_i\}_{i \in [n]}$
- $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$
- Fundamental weights: $\{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_{n-1}, (e_1 + e_2 + \cdots + e_n)/2\}$.

Consider the $(W, W_{[n]\setminus n})$ Coxeter matroid polytopes of type $B_n$. The vertices are the orbits of $(e_1 + e_2 + \cdots + e_n)/2$.

Hence a polytope is a type $B_n$ Coxeter matroid polytope if and only if the vertices are contained in the vertices of the cube $[-1/2, 1/2]^n$ and edge directions are parallel to $\pm e_i \pm e_j$ or $e_i$.
**Exercise:** Visualize some two and three dimensional examples of these polytopes.

# Type $B_n$ Coxeter Matroid Polytopes

## Definition (Type $B_n$ Root Datum)

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- $\Phi = \{\pm e_i \pm e_j\}_{i \neq j \in [n]} \cup \{\pm e_i\}_{i \in [n]}$
- $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$
- Fundamental weights: $\{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_{n-1}, (e_1 + e_2 + \cdots + e_n)/2\}$.

Another example are the $(W, W_I)$ Coxeter matroid polytopes where $n = 4$ and $I = [n]\backslash 2$. Then we are studying polytopes whose vertices are $0, 1, -1$ vectors with exactly two non-zero entries and whose edge directions are parallel to the roots.

# Type $C_n$ Coxeter Matroid Polytopes

## Definition (Type $C_n$ Root Datum)

- $W \cong \{\text{signed permutations of n}\}$
- $\Phi = \{\pm e_i \pm e_j\}_{i \neq j \in [n]} \cup \{\pm 2e_i\}_{i \in [n]}$
- $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\}$
- Fundamental weights:
  $\{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_{n-1}, e_1 + e_2 + \cdots + e_n\}$.

The $(W, W_{[n]\setminus n})$ Coxeter matroid polytopes are polytopes whose vertices are subsets of the $[-1, 1]^n$ cube and whose edge directions are parallel to the roots.

Note the slight difference from the type $B_n$ situation.

# Coxeter Matroid Polytopes

The vertices of a Coxeter matroid polytope are a subset of the orbit $W \cdot \rho_I$ which is in bijection with $W/W_I$.

**Theorem**

*Under this bijection, the vertices of a Coxeter matroid polytope form a Coxeter matroid. Further, every Coxeter matroid arises in this way.*

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**Spooky thought:** Note that the concept of a Coxeter matroid polytope distinguishes between type $B_n$ and type $C_n$ root systems whereas the greedy algorithm Coxeter matroid does not.

# What is the role of $\rho_I$?

A choice of simple system of $\Phi$ gives a choice of a fundamental chamber of $\mathcal{A}$. This will always be a simplicial cone generated by the vectors $\rho_1, \ldots, \rho_n$.

Every $\rho_i$ is the minimal lattice point in the ray corresponding to $\rho_i$. Further, $\rho_I$ is the minimal lattice point in the interior of the cone generated by $\{\rho_i\}_{i \in I}$.

Previously, we saw that every cone of the fundamental chamber is indexed by a parabolic subgroup $W_I$. The point $\rho_{[n]/I}$ gives a minimal interior lattice point in the cone indexed by $W_I$.

# Geometric Intuition

Let $G$ be a simple Lie group. For every sum of fundamental weights $\rho_I$, there is an irreducible representation $V(\rho_I)$ of $G$ corresponding to $\rho_I$.

The representation $V(\rho_I)$ has a vector space decomposition into **weight spaces** indexed by the $\Lambda$-lattice points contained in the convex hull of $W \cdot \rho_I$.

$$V(\rho_I) = \bigoplus_{\alpha \in \Lambda \cap \mathrm{conv}(W \cdot \rho_I)} V_\alpha.$$

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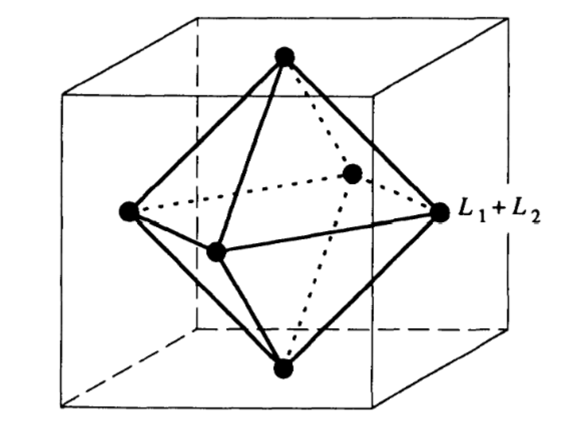
Let $\eta$ be a vector in the weight space corresponding to $\rho_I$. Let $P$ be the subgroup of $G$ that stabilizes this weight space. This gives an embedding of $G/P \to \mathbb{P}(V(\rho_I))$ by $g \mapsto g \cdot \eta$.

This generalizes the Plücker embedding of the Grassmannian.

# Geometric Intution

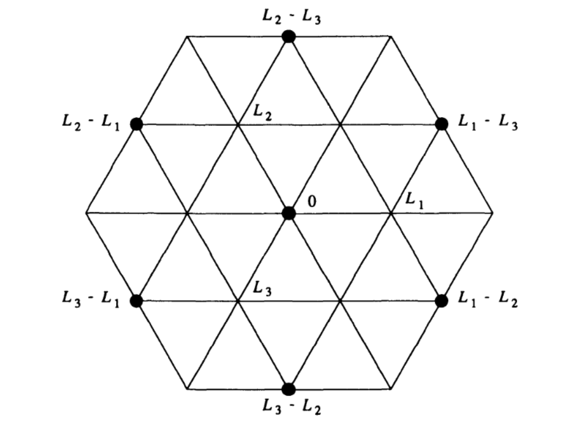
Take $G = SL_4$ and the representation with heighest weight $\rho_2$. Then, $G/P$ corresponds to the Grassmannian.




The weights can be indexed by subsets of size 2.

# Geometric Intution

Take $G = SL_3$ and the representation with heighest weight $\rho_{\{1,2\}}$.




The extremal weights are indexed by elements of $w \in W$ and then we have the center weight.

# Geometric Intuition

Every choice of $w \in W$ induces a stratification of $G/P$ into cells indexed by elements in $W/W_I$ called the **Bruhat decomposition**.

Gelfand and Serganova studied the **thin stratification** which is the simultaneous refinement of the Bruhat decomposition over all choices of $w \in W$.

# Geometric Intuition

Every choice of $w \in W$ induces a stratification of $G/P$ into cells indexed by elements in $W/W_I$ called the **Bruhat decomposition**.

Gelfand and Serganova studied the **thin stratification** which is the simultaneous refinement of the Bruhat decomposition over all choices of $w \in W$.

## Theorem (Gelfand-Serganova)

*Fix $W$ and $W_I$. Consider the embedding of $G/P \mapsto \mathbb{P}(V(\rho_{[n]\setminus I}))$. Two elements $x, y \in G/P$ lie in the same thin stratum if and only if for every coordinate indexed by a weight $\alpha$ in the orbit of $\rho_{[n]\setminus I}$ we have that $p_\alpha(x) = 0$ if and only if $p_\alpha(y) = 0$.*

# Gelfand-Serganova's Definition of Coxeter Matroid Polytopes

The definition of Coxeter matroid polytope I have given disagrees with Gelfand and Serganova's definition slightly.

### Definition (Gelfand-Serganova Coxeter Matroid Polytope)

A $(W, W_I)$ **Coxeter matroid polytope** is a polytope whose vertices lie in the orbit of any point $\omega_I$ in the interior of the face of the fundamental chamber indexed by $W_I$ and whose edge directions are parallel to the roots of $\Phi$.

### Theorem (Borovik)

*The combinatorial type of a Coxeter matroid polytope does not depend on the choice of* $\omega_I$*.*

The version I stated has the advantage of giving a "canonical" choice of $\omega_I$ at the cost(?) of introducing the weight lattice.

# What about the other definitions?

What do we know about the different cryptomorphic definitions of Coxeter matroids?

- **Basis Exchange** - The näive notion fails where bases correspond to the elements of $W/W_I$ that are in the matroid. Not even true in type $A_n$. Mysteriously everything seems to work for the minuscle cases.
- **Flat Partition Axiom** - What is a flat?
- **Rank Function** - There is a notion of Coxeter submodular functions which is close to giving an answer!
- **Greedy Algorithm** - Our main approach. However, some of the geometric connections to the flag varieties fail!
- **Circuit Axiom** - What is a circuit?
- **Matroid Polytope** - We have two slightly different definitions. Either way, we understand this best.