Coxeter Matroids
Sections 3.1–3.5

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A **symplectic matroid of rank** k is a collection $\mathcal{B} \subseteq J_k$ such that for every $w \in BC_n$, there is a unique maximal element of $\mathcal{B}$ with respect to the ordering $\leq^w$. The sets in $\mathcal{B}$ are the **bases**. A symplectic matroid of rank n is a **Lagrangian matroid**.

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Also,

$$\Pi = \{2\epsilon_1, \epsilon_2 - \epsilon_1, \ldots, \epsilon_n - \epsilon_{n-1}\}$$

is the set of simple roots.

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Theorem (Gelfand–Serganova Theorem for Symplectic Matroids)

Let $\mathcal{B} \subseteq J_k$, and let Δ be the convex hull of $\{\delta_A : A \in \mathcal{B}\}$. Then $\mathcal{B}$ is a symplectic matroid if and only if all edges of Δ are parallel to roots in Φ.

Isotropic Subspaces

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The **standard symplectic space** is the vector space V with basis $E = \{e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*}\}$ and an anti-symmetric bilinear form $(\cdot, \cdot)$ satisfying $(e_i, e_j) = 0$ for all $i, j \in J$ with $i \neq j^*$ and $(e_i, e_{i^*}) = -(e_{i^*}, e_i) = 1$ for all $i \in [n]$.

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A subspace U of V is **isotropic** if $(u, v) = 0$ for all $u, v \in U$.

Lemma: A subspace U of V is isotropic if and only if it can be represented by a $k \times 2n$ matrix (A, B) such that AB^t is symmetric.

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Given a $k \times 2n$ matrix (A, B) with columns indexed by J, consider the collection $\mathcal{B} \subseteq J_k$ of admissible k-subsets K such that the $k \times k$ minor of (A, B) with column set K is nonzero.

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If $\Lambda \in \mathrm{GL}_n$ is diagonal, then (A, B) and $(A\Lambda^{-1}, B\Lambda)$ represent the same symplectic matroid.

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Theorem: Let $k = m + \ell$, and let $\mathcal{B} \subseteq J_k$ be m-homogeneous. Then $\mathcal{B}$ is a symplectic matroid if and only if $\mathrm{flag}(\mathcal{B}) := \{\mathrm{flag}(A) : A \in \mathcal{B}\}$ is a (type-A) flag matroid of rank $(m, n - \ell)$.

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Example: Let $m = 2$, $\ell = 1$, (so $k = 3$), and $n = 4$.
Let $\mathcal{B} = \{124^*, 123^*, 13^*4, 12^*4, 234^*, 2^*34, 23^*4\}$. Then $\mathrm{flag}(\mathcal{B})$ is $\{(12, 123), (12, 124), (14, 124), (14, 134), (23, 123), (34, 134), (24, 124)\}$.
The collection $\mathcal{B}$ is a 2-homogeneous symplectic matroid, and $\mathrm{flag}(\mathcal{B})$ is a flag matroid of rank $(2, 3)$.

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Theorem

Let $\mathcal{B}$ be a symplectic matroid of rank $k = m + \ell$ represented by a $k \times 2n$ matrix (A, B). The following are equivalent:

1. *$\mathcal{B}$ is m-homogeneous.*
2. $\operatorname{rank}(A) = m$ *and* $\operatorname{rank}(B) = \ell$.
3. *$\mathcal{B}$ may be represented by a matrix of the form* $\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix}$, *where Y is $m \times n$, Z is $\ell \times n$, and $YZ^t = 0$.*
4. *$\mathcal{B}$ is m-homogeneous, the constituent of* $\operatorname{flag}(\mathcal{B})$ *of rank m is represented by* $\operatorname{rowsp}(Y)$, *the constituent of* $\operatorname{flag}(\mathcal{B})$ *of rank $n - \ell$ is represented by* $(\operatorname{rowsp}(Z))^{\perp}$, *and* $\operatorname{rowsp}(Y) \subseteq (\operatorname{rowsp}(Z))^{\perp}$.