Chap 1|

- If M representable by subspace $\mathcal{U}$, rk fcn can be recovered by looking @ dim of intersection of $\mathcal{U}$ w/ coord subspaces |this is more or less why intersection of Schub cells give matroid strata, I think|

Recall $\mu: Gr_{k,n} \longrightarrow \mathbb{R}^n$ defined as $\mu_i(p) = \dfrac{\sum_{J \ni i} |\Delta_J(p)|^2}{\sum_{J \in \binom{[n]}{k}} |\Delta_J(p)|^2}$.

Thm 1.2.1: $\overline{H \cdot p}$ = closure of torus orbit of p. $\mu(\overline{H\cdot p}) = \mathrm{conv}(e_I \mid I \text{ basis of } M_p)$

§1.3: Maximality property|

Defn: $A, B \in \binom{[n]}{k}$, $w \in S_n$. $A \leq^w B \Leftrightarrow w^{-1}(A) \leq w^{-1}(B)$ "Gale ordering of $\binom{[n]}{k}$ induced by w"

↳ if $w = w_1 \cdots w_n$ (one-line notation) then $w_1 <^w w_2 <^w \cdots <^w w_n$.

Thm 1.3.1: $\mathcal{B} \subseteq \binom{[n]}{k}$ matroid iff satisfies Maximality Property (MP):

$$\forall w \in S_n,\ \exists!\ A \in \mathcal{B} \text{ s.t. } B \leq^w A\ \forall\ B \in \mathcal{B}.$$

Pf: Use Basis Exchange

§1.4: Increasing Exchange Property|

IEP: $\mathcal{B} \subseteq \binom{[n]}{k}$, $A_1 \neq A_2 \in \mathcal{B}$, $w \in S_n$. $\exists$ transposition $t = (a\ b)$ w/ $a <^w b$ s.t. one of A_1, A_2, say A_i, contains a, does not contain b & $tA_i = A_i \setminus a \cup b \in \mathcal{B}$.

Thm 1.4.1: MP $\Leftrightarrow$ IEP

Pf: Usual exchange $\Rightarrow$ IEP (by contradiction) & IEP $\Rightarrow$ MP (by contradiction)

§1.5: Sufficient Systems of Exchanges|

- $\mathbb{R}^n$ w/ ordered orthonormal basis ϵ_i, scalar prod. $\langle \cdot, \cdot \rangle$.

$\Phi := \{\epsilon_i - \epsilon_j \mid i \neq j\}$

Defn: $\Psi \subseteq \Phi$ a covering set if the system of ineq.

$$\langle x, \psi \rangle < 0 \quad \forall\ \psi \in \Psi$$

has no soln (i.e. half-spaces $\{x \in \mathbb{R}^n \mid \langle x, \psi \rangle \geq 0\}$ cover $\mathbb{R}^n$)

Note: If $\Psi \subseteq \Phi$ satisfies $\sum_{\psi \in \Psi} \psi = 0$, then Ψ a covering set. (since then $\forall x \in \mathbb{R}^n$ $\sum_\psi \langle x, \psi \rangle = 0 \Rightarrow$ at least one $\langle x, \psi \rangle \geq 0$)

- $A \in \binom{[n]}{k}$, $t = (a, b)$ transposition that "moves" A (i.e. $a \in A$, $b \notin A$). ⇝ assoc. exchange $A \mapsto tA$ to root $\psi_{ab} = \psi_t = e_b - e_a$

Defn: $A \neq B \in \binom{[n]}{k}$. System of exchanges $\underbrace{s_1, \dots, s_\ell}_{\text{move } A}, \underbrace{t_1, \dots, t_m}_{\text{move } B}$ is <u>sufficient</u> if the set of roots $\{\psi_{s_i}\} \cup \{\psi_{t_i}\}$ is covering.

Thm: $\mathcal{B} \subseteq \binom{[n]}{k}$. Sup. $\forall\, A \neq B \in \binom{[n]}{k}$ $\exists$ sufficient system of exchanges $s_1, \dots, s_\ell, t_1, \dots, t_m$ s.t. $s_i A \in \mathcal{B}$ & $t_j B \in \mathcal{B}$ $\forall i, j$. (We say "$\mathcal{B}$ admits a sufficient system of exchanges")
Then $\mathcal{B}$ satisfies IEP (& thus is matroid).

Pf: By contradiction. If IEP not satisfied by $A \neq B \in \mathcal{B}$, (say it fails for $w \in S_n$) then let $s_1, \dots, s_\ell$ be all transpositions moving A to another elt of $\mathcal{B}$ & define $t_1, \dots, t_m$ analogously.
$R = \{\psi_{s_i}\} \cup \{\psi_{t_i}\}$ is not covering, because

$$\left\langle \sum_{i=1}^{n} w^{-1}(i) e_i, \psi \right\rangle < 0 \quad \forall\, \psi \in R. \text{ (precisely because IEP fails for } w, A, B) \quad \blacksquare$$

Strong exchange prop: (SEP) $\mathcal{B} \subseteq \binom{[n]}{k}$ satisfies SEP if $\forall A \neq B \in \mathcal{B}$, $\exists\, a \in A \setminus B$ & $b \in B \setminus A$ s.t. $A \setminus a \cup b \in \mathcal{B}$ & $B \setminus b \cup a \in \mathcal{B}$.

Thm: If $\mathcal{B} \subseteq \binom{[n]}{k}$ satisfies SEP, then $\mathcal{B}$ is the bases of a matroid.

Pf: $A \neq B \in \mathcal{B}$. Take a, b from SEP. Then ψ_{ab}, ψ_{ba} are sufficient system of exchanges ($\psi_{ab} + \psi_{ba} = 0$) so $\mathcal{B}$ satisfies IEP.

§1.6: Matroids as maps

$\mathcal{B}$ matroid of rk k on $[n]$.

$\rightsquigarrow$ map $\mu : S_n \longrightarrow \binom{[n]}{k}$
$\qquad w \longmapsto A \in \mathcal{B}$ max'l wrt $<^w$.

Note: μ satisfies $\mu(u) \leq^w \mu(w)$ $\forall\, u, w \in S_n$. "matroid inequality"

Also, $\mu(S_n) = \mathcal{B}$ (preimage of $A = \{a_1 < \dots < a_k\} \in \mathcal{B}$ is any $w \in S_n$ like $w = \underbrace{\dots}_{\text{whatever}} a_1 \cdots a_k$)

Note: The image of every map $S_n \to \binom{[n]}{k}$ satisfying the matroid ineq. is a matroid.