Large induced acyclic and outerplanar subgraphs of low-treewidth planar graphs

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Abstract. Albertson and Berman conjectured that every planar graph has an induced forest on half of its vertices. The best known lower bound, due to Borodin, is that every planar graph has an induced forest on two fifths of its vertices. In a related result, Chartran and Kronk, proved that the vertices of every planar graph can be partitioned into three sets, each of which induce a forest.

We show tighter results for planar graphs of low treewidth. We show that the Albertson-Berman conjecture holds, and is tight, for planar graphs of treewidth 3 (and, in fact, for any graph of treewidth at most 3). We show that every 2-outerplanar graph has an induced outerplanar graph on at least two-thirds of its vertices. We also show that every 2-outerplanar graph has an induced forest on at least half the vertices by showing that its vertices can be partitioned into two sets, each of which induces a forest.

Keywords: Induced forest, planar graphs, treewidth, outerplanarity

1 Introduction

For many optimization problems, finding subgraphs with certain properties is a key to developing algorithms with efficient running times or bounded approximation ratios. For example, balanced separator subgraphs support the design of divide-and-conquer algorithms for minor-closed graph families [17, 18, 13, 16] and large subgraphs of low-treewidth[3] support the design of approximation schemes, also for minor-closed graph families [4, 8, 12]. In the area of graph drawing, one often starts by drawing a subgraph that is somehow easier to draw than the entire graph (such as a planar graph or a tree) and then adding in remaining graph features [5]; the larger the subgraph, the bigger the head-start for drawing and the more structure the subgraph has, the easier the subgraph will be to draw.

In this paper we are concerned with finding large induced subgraphs, in particular large induced forests and large induced outerplanar subgraphs, of input planar graphs. We are motivated both by the intrigue of various conjectures

[3] Formal definitions of graph theoretic terms will be given at the end of this section.

in graph theory but also by the impact that graph theoretic results have on the design of efficient and accurate algorithms. In particular, many algorithms that are specifically designed for planar graphs rely on deep graph theoretic properties of planar graphs [22, 23].

1.1 Large induced forests of planar graphs: known results

Albertson and Berman conjectured that every planar graph has an induced forest on at least half of its vertices [2]; K_4 illustrates that this would be the best possible lower bound. A proof of the Albertson-Berman Conjecture would, among other things, would provide an alternate proof, avoiding the 4-Color Theorem, that every planar graph has an independent set with at least one-quarter of the vertices.

The best-known lower bound toward the Albertson-Berman Conjecture has stood for 40 years: Borodin showed that planar graphs are *acyclically 5-colorable* (i.e. have a 5-coloring, every two classes of which induce a forest), thus showing that every planar graph has an induced forest on at least two-fifths of its vertices [6]. This is the best lower bound achievable toward the Albertson-Berman Conjecture via acyclic colorings as there are planar graphs which do not have an acyclic 4-coloring (for example $K_{2,2,2}$ or the octahedron).

The Albertson-Berman Conjecture has been proven for certain subclasses of planar graphs. Hosono showed that outerplanar graphs have induced forests on at least two-thirds of the vertices [15] and Salavatipour shows that every triangle-free planar graph on n vertices has an induced forest with at least $\frac{17n+24}{32}$ vertices [20], later improved to $\frac{6n+7}{11}$ by Dross, Montassier and Pinou [10]. In bipartite planar graphs, the best bound on the size of the largest induced forest is $\frac{4n}{7}$ by Wang, Xie and Yu [21].

One direction toward proving the Albertson-Berman Conjecture is to partition the vertices of graph G into $a(G)$ sets such that each set induces a forest; $a(G)$ is the *vertex arboricity* of G. This implies that G has an induced forest with at least $1/a(G)$ of its vertices. Chartran and Kronk first proved that all planar graphs have vertex arboricity at most 3 [7]. Raspaud and Wang proved that $a(G) \leq 2$ if G is planar and either G has no 4-cycles, any two triangles of G are at distance at least 3, or G has at most 20 vertices; they also illustrated a 3-outerplanar graph on 21 vertices with vertex arboricity 3 [19]. Yang and Yuan [1] proved that $a(G) \leq 2$ if G is planar and has diameter at most 2.

1.2 Outline of our results

In this paper, we prove the Albertson-Berman Conjecture for graphs of treewidth at most 3 by showing, more generally, that graphs of treewidth k have acyclic $(k+1)$-colorings (Section 2). We further show that 2-outerplanar graphs, which have treewidth at most 5, have vertex arboricity 2, thus showing that they satisfy the Albertson-Berman Conjecture and closing the gap for planar graphs with vertex arboricity 2 versus 3 left by Raspaud and Wang's work (Section 3). Finally we show that every 2-outerplanar graph has an induced outerplanar graph on at least two-thirds of its vertices and propose a few related conjectures (Section 4).

1.3 Definitions

We use standard graph theoretic notation [9]. In this paper, all graphs are assumed to be finite and simple (without loops or parallel edges). $G[S]$ denotes the *induced subgraph* of graph G on vertex subset S: the graph having S as its vertices and having as edges every edge in G that has both endpoints in S. Equivalently, $G[S]$ may be constructed from G by deleting every vertex and incident edges that is not in S. We use $d_H(v)$ to denote degree of vertex v in graph H and $|H|$ to denote the number of vertices of graph H.

Block-Cut Tree. A block of a graph G is a maximal two-connected component of G. A block-cut tree $\mathcal{T}$ of a connected graph G is a tree where each vertex of $\mathcal{T}$ corresponds to block and there is an edge between two vertices X, Y of $\mathcal{T}$ if two blocks X and Y share a common vertex or are incident to a common edge.

Planar graphs. A graph G is *planar* if it can be drawn (embedded) in the plane without any edge crossings. Although a planar graph may have many different embeddings, throughout this paper, we will assume that we are given a fixed embedding of the graph. A *face* of a planar graph is connected region of the complement of the image of the drawing. There is one *infinite* face, which we denote by f_∞. We denote the boundary of f_∞, which is the boundary of G, by ∂G. We say that a vertex v is enclosed by a cycle C if every curve from the image of v to an infinite point must cross the image of C.

Planar duality. Every planar graph G has a corresponding dual planar graph G^*: the vertices of G^* correspond to the faces of G and the faces of G^* correspond to the vertices of G; an edge of G^* connects two vertices of G^* if the corresponding faces of G share an edge (in this way the edges of the two graphs are in bijection).

Outerplanarity. A planar graph G with a given embedding is *outerplanar* (or 1-*outerplanar*) if all vertices are in ∂G. A planar graph is k-*outerplanar* for $k > 1$ if deleting the vertices in ∂G results in a $(k-1)$-outerplanar graph. A k-outerplanar graph has a natural partition of the vertices into k *layers*: L_1 is the set of vertices in ∂G; L_i is the set of vertices in the boundary of $G \setminus \cup_{j<i} L_j$. We denote $G(V, E)$ by $G(L_1, \ldots, L_k; E)$ if G is k-outerplanar. For a 2-outerplanar graph, we define the *between degree* of a vertex $v \in L_i$ to be the number of adjacent vertices in $L_j, j \neq i$.

Facial Block. Let $\mathcal{C}$ be the set of facial cycles bounding finite faces of $G[L_1]$. For each $C \in \mathcal{C}$, let S_C be the set of vertices enclosed by Cin G. Then we call the graph $G[C \cup S_C]$ a facial block of G.

Treewidth. A graph has treewidth k if k is the least integer such that G is the subgraph of a k-tree. A graph G is a k-*tree* if there is a vertex v of degree k in G that is incident to the vertices of a k-clique such that deleting v results in a k-tree *or* G is the complete graph on k vertices. Note that k-outerplanar graphs have treewidth at most $3k - 1$ [11].

2 Treewidth-k graphs admit acyclic $(k + 1)$-colorings

An m-*coloring* of a graph G is an assignment of one of m colors to each vertex such that no adjacent vertices are the same color. An *acyclic* m-coloring of G is a proper m-coloring in which the union of any two color classes induces an acyclic subgraph (or *forest*).

Lemma 1. *Graphs of treewidth k admit acyclic $(k+1)$-colorings.*

Proof. Let H be such a graph and let G be a supergraph of H on the same set of vertices such that G is a k-tree. We show that G has an acyclic $(k+1)$-coloring; this coloring is an acyclic $(k+1)$-coloring of H.

We proceed by induction on the number of vertices $|V|$ of G. When $|V| \leq k$, any coloring with at most $|V|$ colors is acyclic.

Suppose $|V| = n$. By the definition of a k-tree, there exists $v \in V$ such that $d(v) = k$ and $G - v$ is a k-tree. $G - v$ has an acyclic $(k + 1)$-coloring by the inductive hypothesis. The neighbors of v induce a k-clique in G and thus are colored k distinct colors in any proper coloring. Color v the remaining color, say i. This coloring is acyclic, v has only one edge with a vertex colored j for $i \neq j$. □

The two largest color classes of a treewidth-k graph as guaranteed by this lemma give the following:

Corollary 1. *Every n-vertex graph of treewidth k has an induced forest on at least $\frac{2n}{k+1}$ vertices.*

This corollary implies the Albertson-Berman Conjecture for graphs of treewidth 3, and, as K_4 has treewidth 3, the conjecture is tight for planar graphs of treewidth 3. For planar graphs of treewidth 2, which includes outerplanar graphs, the corollary implies the existence of an induced forest on $\frac{2}{3}$ of the vertices (meeting Hosono's result). This bound is again tight, as the union of disjoint triangles is outerplanar.

3 2-outerplanar graphs have vertex-arboricity 2

In this section, we prove:

Theorem 1. *If G is a 2-outerplanar graph, then the vertex arboricity of G is 2: $a(G) \leq 2$.*

We call a set of vertex-disjoint induced forests of G *induced p-forests* if their vertices partition the vertex set of G. We consider a counterexample graph G of minimal order. By studying the structure of this minimal counterexample, we will derive a contradiction. Let e be an edge that is not in G. We observe:

Observation 2 *If $a(G \cup \{e\}) \leq 2$, then $a(G) \leq 2$.*

Observation 2 allows us to assume w.l.o.g. that G is connected (by adding edges between components while maintaining 2-outerplanarity) and that G is a disk triangulation, i.e., that every face except the outer face of G is a triangle (by adding edges between layers of vertices while maintaining 2-outerplanarity).

Claim 3 *Every vertex in G has degree at least 4.*

Proof. Suppose G has a vertex v of degree at most 3. Since G is a minimal order counterexample and $G-v$ is a 2-outerplanar graph, $a(G-v) = 2$. Let F_0 and F_1 be two induced p-forests of $G-v$. Since v has at most 3 neighbors in G, one of F_0 or F_1, w.l.o.g. say F_0, contains at most one of these neighbors. Therefore $F_0 \cup v$ is a forest of G and $F_0 \cup \{v\}, F_1$ are two induced p-forests of G, contradicting that G is a counterexample. □

Let L_1, L_2 be the bipartition of the vertices of G into layers. Let B be a facial block of G that shares at most one edge or at most one vertex with other facial blocks of G. We can choose B as the facial block which is enclosed by a cycle bounding a finite face of G that has at most one edge not in ∂G. In the former case, let e_B be the shared edge; in the later case, let e_B be either edge in ∂B incident to the shared vertex. If $B \equiv G$, let e_B be any edge of G. Denote $L_2^B = L_2 \cap V(B)$. We have:

Claim 4 *$|L_2^B| \geq 2$.*

Proof. If $|L_2^B| = 0$, then B is a triangle since G is a disk-triangulation and vertices have degree at least 4. Then, the vertex of B that is not an endpoint of e_B has degree 2 in G, contradicting Claim 3. If $L_2^B = \{v\}$, By Claim 3, v has at least four neighbors in L_1 and thus, at least one neighbor u of v in L_1 is not an endpoint of e_B. Then the degree of u in G is 3, contradicting Claim 3. □

Claim 5 *Let $v \in L_2^B$ have between degree at least 3. Then, either v is a cut vertex of $G[L_2^B]$ or v is adjacent to both endpoints of e_B.*

Proof. Let v_1, v_2, v_3 be neighbors of v in ∂B in clockwise order around v. Let $\partial B[v_i, v_j]$ be the clockwise segment of ∂B from v_i to v_j, $i \neq j$. We define $C_{ij} = \partial B[v_i, v_j] \cup \{vv_i, vv_j\}$, which is a cycle of B. Since v is non-cut, at most one cycle of $\{C_{12}, C_{23}, C_{31}\}$ encloses a vertex of L_2^B, say C_{31}. Thus, v_2 is only adjacent to v and two other neighbors, say v'_1, v'_3, of ∂B. Since C_{12} and C_{23} enclose no vertex of L_2^B, vv'_1v_2 and $vv_2v'_3$ are faces of G. If neither $v'_1v_2 = e_B$ nor $v_2v'_3 = e_B$, then $d_G(v_2) = 3$, contradicting Claim 3. □

Suppose $v \in L_2^B$ is such that $d_{G[L_2^B]}(v) = 1$. By Claim 3, $d_G(v) \geq 4$ so v has between degree at least 3. Thus, by Claim 5, we have:

Observation 6 *If there exists $v \in L_2^B$ such that $d_{G[L_2]}(v) = 1$, then v must be adjacent to both endpoints of e_B.*

Let x_B, y_B be endpoints of e_B. Since G is a triangulation, there is a vertex $v \in L_2^B$ such that vx_By_B is a face of G. We call v the *separating vertex* of B.

Claim 7 *If $v' \neq v$ is a vertex in L_2^B that is adjacent to both endpoints of e_B, then, v' is a cut vertex of L_2^B.*

Proof. We will prove that v' has at least one neighbor in L_2^B inside the triangle $v'x_By_B$ and at least one neighbor in L_2^B outside the triangle $v'x_By_B$; thus v' is a cut vertex of L_2^B.

By planarity, the triangle $v'x_By_B$ encloses v. Let $C_{vv'} = \{v, x_B, v', y_B\}$ which is a cycle of G. Since G is a disk triangulation and the edge x_B, y_B is embedded outside $C_{vv'}$, there must be an edge or a path inside $C_{vv'}$ connecting v and v'. Thus v' has at least one neighbor in L_2^B inside the triangle $v'x_By_B$.

We note that the cycle $\{\partial B \setminus e_B\} \cup \{vx_b, vy_B\}$ must enclose at least one vertex of L_2^B since otherwise, the neighbor of v on $\{\partial B \setminus e_B\}$ that is not endpoints of e_B has degree 3; contradicting Claim 3. Thus, v has at least one neighbor in L_B^2 outside the triangle $v'x_By_B$ as desired. □

Since every cut vertex of L_2^B has degree at least 2 in $G[L_2^B]$, by Claim 7 and Observation 6, we have:

Observation 8 *Only separating vertex v of B can have $d_{G[L_2]}(v) = 1$.*

If the block-cut tree of $G[L_2^B]$ has at least two vertices, let K be a leaf block of $G[L_2^B]$ that does not contain the separating vertex of B. In this case, by Observation 8, $|K| \geq 3$. Otherwise, let $K = G[L_2^B]$. We refer the cut vertex of K if $K \neq G[L_2^B]$ in the former case and the separating vertex of B in the latter case as the *separating vertex* of K. By Claim 5, we have:

Observation 9 *Non-separating vertices of K have between degree at most 2.*

We call a triangle abc of K a *critical triangle* with top c if $d_K(c) = 2$ and c is non-separating. By Observation 9 and Claim 3, c has exactly two neighbors in L_1, that we denote by d, e (See Figure 1). Since G is a disk triangulation, two edges da and eb are edges of G.

Claim 10 *Vertices d and e, each has degree at least 5.*

Proof. Neither d nor e has degree less than 4 by Claim 3. For contradiction, w.l.o.g, we assume that $d_G(d) = 4$. Let f be a neighbor of d on L_1 that is different from e (See Figure 1). Since G is a disk triangulation, $af \in E(G)$. Let G' be the graph obtained from G by contracting fd and dc and removing parallel edges. Then G' is a minor of G (and so is 2-outerplanar) with fewer vertices. Let F_0, F_1 be two induced p-forests of G' that exist by the minimality of G. Without loss of generality, we assume that $f \in F_0$. We have two cases:

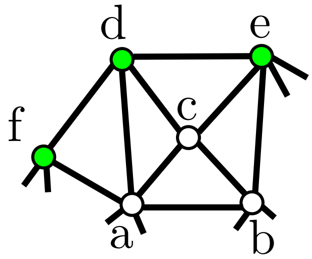

Fig. 1. The critical triangle abc and two neighbors d, e of c in L_1. Hollow vertices are in L_2.

1. If $b \in F_0$, then $a, e \in F_1$. If $bf \notin G$, adding c, d to F_0 does not destroy the acyclicity of F_0 in G. Thus, $F_0 \cup \{c, d\}, F_1$ are two induced p-forests of G. If $bf \in G$, bf separates a from e so a and e are in different trees in F_1. Thus, $F_0 \cup \{c\}, F_1 \cup \{d\}$ are two induced p-forests of G.
2. Otherwise, $b \in F_1$. We have three subcases:
   (a) If a, e are both in F_0, then $F_0, F_1 \cup \{c, d\}$ are two induced p-forests of G.
   (b) If $a, e \in F_1$, then, $F_0 \cup \{c, d\}, F_1$ are two induced p-forests of G.
   (c) If a, e are in different induced p-forests of G', then, $F_0 \cup \{c\}, F_1 \cup \{d\}$ are two induced p-forests of G.

In each case, the resulting p-forests contradicts that G is a minimal order counter example. □

Claim 11 $|K| \geq 4$.

Proof. If $K \neq G[L_2^B]$, as noted in the definition of K, $|K| \geq 3$, If $K = G[L_2^B]$, then by Claim 4, $|K| \geq 2$ and by Observation 8, $|G| \geq 3$. Suppose that $|K| = 3$. Then, K is a triangle. Let u, w be two neighbors of the cut vertex v in K. Then, wuv is a critical triangle with top u (or w). By Claim 3 and Observation 9, u and w both have between degree 2. Thus, u and w have a common neighbor on L_1 which therefore has degree 4, contradicting Claim 10. □

Suppose that a and b of the critical triangle abc with top c of K have a common neighbor f in L_2. We have:

Claim 12 *If fa (resp. fb) is in $\partial G[L_2^B]$, then a (resp. b) must be the separating vertex.*

Proof. For a contradiction (and w.l.o.g), we assume that $fa \in \partial G[L_2^B]$ and a is non-separating. See Figure 2. Let G' be the graph obtained from G by contracting ac and ce and removing parallel edges. Then, G' is a minor of G (and so is 2-outerplanar) with fewer vertices. Let F_0, F_1 be two induced p-forests of G', which are guaranteed to exist by the minimality of G. Without loss of generality, we assume that $f \in F_0$. We consider two cases:

1. If $e \in F_0$, then d, b are in F_1. If edge $fe \notin G$, then, $F_0 \cup \{a, c\}, F_1$ are two induced p-forests of G. If $fe \in G$, fe separates d from b so d and b are in different trees of F_1. Thus, $F_1 \cup \{c\}, F_1 \cup \{a\}$ are two induced p-forests of G.
2. Otherwise, $e \in F_1$. We have three subcases:
   (a) If b, d are both in F_0, then $F_0, F_1 \cup \{a, c\}$ are two induced p-forests of G.
   (b) If b, d are both in F_1, then $F_0 \cup \{a, c\}, F_1$ are two induced p-forests of G.
   (c) If b, d are in different forests of G', then, $F_0 \cup \{c\}, F_1 \cup \{b\}$ are two induced p-forests of G.

In each case, the resulting p-forests contradicts that G is a minimal order counter example. □

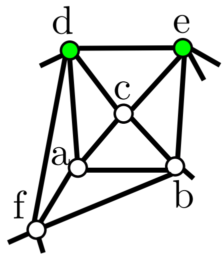

Fig. 2. The critical triangle abc with edge $fa \in \partial G[L_2]$. Hollow vertices are in L_2.

If the edge fb is shared with another critical triangle fbg with top g, then we call $\{abc, bfg\}$ a *pair of critical triangles.* See Figure 3.

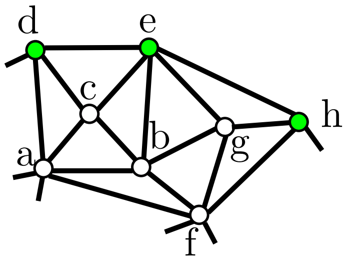

Fig. 3. A pair of critical triangles abc and bfg. Hollow vertices are in L_2

Claim 13 *If there exists a pair of critical triangles abc and bfg in K, then b must be the separating vertex of K.*

Proof. Note that neither c nor g can be the separating vertex by definition of critical triangles. Suppose for contradiction that b is non-separating. Let d, e

be two neighbors of c as defined above and i and h be the neighbors of g in L, as argued. We first argue that $i \equiv e$. Suppose otherwise. Since G is a disk triangulation, ec, eb, ig, ih are edges of G. Since b is non-separating, the triangle ebi is empty; e would have degree 4, contradicting Claim 10.

We also note that $h \neq d$ (for otherwise, G would have parallel edges connecting d and e) and $hf \in E(G)$. See Figure 3. Let G' be the graph obtained from G by contracting ec, eb, eg and eh and removing parallel edges. Thus, G' is a minor of G with fewer vertices. By minimality, G' has two induced p-forests F_0, F_1. Without loss of generality, we assume that $a \in F_0$. We will reconstruct two induced p-forests of G by considering two cases:

1. If $h \in F_0$, then $d, f \in F_1$. If edge $ah \in G$, then, by planarity, d and f are in different trees of F_1. Thus, $F_1 \cup \{e, b\}$ has no cycle that implies $F_1 \cup \{e, b\}, F_0 \cup \{c, g\}$ are two induced p-forests in G. Otherwise, $F_0 \cup \{b, e\}$ has no cycle. Thus, $F_0 \cup \{b, e\}, F_1 \cup \{c, g\}$ are two induced p-forests in G.
2. Otherwise, $h \in F_1$. We have four subcases:
   (a) If d, f are both in F_1, then $F_0 \cup \{c, e, g\}, F_1 \cup \{b\}$ are two induced p-forests of G.
   (b) If d, f are both in F_0, then $F_0 \cup \{e\}, F_1 \cup \{b, c, g\}$ are two induced p-forests of G.
   (c) If $d \in F_0, f \in F_1$, then $F_0 \cup \{e, g\}, F_1 \cup \{b, c\}$ are two induced p-forests of G.
   (d) If $d \in F_1, f \in F_0$, then $F_0 \cup \{c, g\}, F_1 \cup \{b, e\}$ are two induced p-forests of G.

Thus, in all cases, the resulting induced p-forests contradict that G is a counterexample. □

We are now ready to complete the proof of Theorem 1 by considering a triangle of K of $G[L_2^B]$, say uvw, containing the separating vertex v of K and has the most edges in common with ∂K. Then, uvw contains at least one edge in ∂K. We note that $K^* \setminus (\partial K)^*$ where $(\partial K)^*$ is the dual vertex of the infinite face of K, is a tree that we denote by T_K^*. We root T_K^* at the vertex corresponding to the triangle uvw. Consider the deepest leaf $x^* \in T_K^*$ and its parent y^*. Let abc be the triangle corresponding to x^* such that the dual edge of ab is x^*y^*. Then $d_K(c) = 2$. Since $K \geq 4$, $abc \not\equiv uvw$ and thus, it is a critical triangle with top c. Let abf be the triangle that corresponds to y^*. Note here it may be that $abf \equiv uvw$. We have three cases:

1. If $d_{T_K^*}(y^*) = 1$, then $abf \equiv uvw$. Thus, two edges fa, fb are both in ∂K but only one of the two vertices a, b can be the separating vertex of K. This contradicts Claim 12.
2. If $d_{T_K^*}(y^*) = 2$, then exactly one of two edges $af, bf \in \partial K$; w.l.o.g, we assume that $bf \in \partial K$. Then, by Claim 12, b must be the separating vertex of K. Thus, only two triangles abc and abf contain the separating vertex. Since uvw is the triangle containing the separating vertex with most edges in ∂K, $uvw \equiv abc$, contradicting our choice of triangle abc.

3. Otherwise, we have $d_{T_K^*}(y^*) = 3$. Then, none of $\{ab, bf, af\}$ is in ∂K, so $abf \not\equiv uvw$. Let z^* and t^* be another two neighbors of y^* in $d_{T_K^*}$ with t^* as the parent of y^*. Then, x^* and z^* have the same depth. By our choice of x^*, z^* must also be a leaf. Thus, the triangle, say bfg, corresponding to z^* is critical. Thus $\{abc, bfg\}$ is a pair of critical triangles. Since t^* is the parent of y^*, b cannot be the separating vertex of K, contradicting Claim 13.

This completes the proof of Theorem 1. □

4 2-outerplanar graphs have large induced outerplanar graphs

In this section, we prove:

Theorem 14. *Let G be a 2-outerplanar graph on n vertices. G has an induced outerplanar subgraph on at least $\frac{2n}{3}$ vertices whose outerplanar embedding is induced from G.*

Let L_1, L_2 be the partition of G into layers. Note that $\partial G[L_i]$ is a cactus graph (every edge is in at most 1 cycle). As in Section 3, we assume w.l.o.g that G is connected and a disk triangulation. This gives us:

Observation 15 *If uv is an edge in $\partial G[L_2]$, then there exists $w \in L_1$ such that uvw is a face.*

Observation 16 *The between degree of every vertex in L_2 is at least 1.*

Lemma 2. *If $v \in L_2$ has between degree 1, then it is incident to exactly two edges in $\partial G[L_2]$.*

Proof. Let u be v's neighbor in L_1. Since G is a disk triangulation, there exist two triangular faces, say xuv and yuv, containing the edge uv. As the between degree of v is 1, x and y are in L_2, and the edges xv and yv are in $\partial G[L_2]$. Therefore, v is incident to at least two edges in $\partial G[L_2]$.

Suppose for the sake of contradiction that v is incident to more than two edges in $\partial G[L_2]$. Let w be a neighbor of v such that $w \notin \{x, y\}$. Then by Observation 15, there exists $s \in L_1$ such that vws is a face. Since $w \notin \{x, y\}$ and G is simple, $s \neq u$. This implies v has between degree at least 2; contradicting that v's between degree is 1. □

Lemma 3. *If a facial block B contains a vertex in L_2, then endpoints of any edge $uv \in \partial B$ are adjacent to a common vertex in L_2^B.*

Proof. Since G is a triangulation, there is a vertex $w \in B$ such that uvw is a triangular face; thus uvw contains no vertex of L_2. Suppose that $w \in \partial B$, then uvw is an induced cycle of $G[L_1]$. By the definition of facial blocks, ∂B bounds a finite face of $G[L_1]$; thus is an induced cycle of $G[L_1]$. Thus, $uvw \equiv \partial B$; contradicting that B contains a vertex in L_2. □

Lemma 4. *There exists a matching $M \subseteq \partial G[L_2]$ with the following property:*

$$\text{If } v \in L_2 \setminus V(M) \text{ then } v \text{ has between degree at least 2.} \quad (1)$$

Proof. Let L be the set of vertices of between degree 1 in L_2. We proceed by strong induction on $|L|$. If $|L| = 0$, any matching $M \subseteq \partial G[L_2]$ has property (1).

Let $v \in L$ and $u \in L_2 \setminus L$ be vertices such that $uv \in \partial G[L_2]$; v exists by Lemma 3. By Lemma 2, v has exactly one other neighbor w such that $vw \in \partial G[L_2]$. Contract uv and wv to v and delete parallel edges and loops; let the resulting graph be G'. Since v now has between degree at least 2, the number of vertices of between degree 1 in G' is strictly less than $|L|$. Therefore, by the inductive hypothesis, there exists a matching $M' \subseteq \partial G'[L_2]$ with property (1). Now, consider M' as a matching in $\partial G[L_2]$.

If v is not covered by M' in G', then u, v, w are not covered by M' in G. Then, $M' \cup \{vw\}$ is a matching and has property (1), since u has between degree at least 2 as argued above.

If $vx \in M'$ for some $x \in G'$, then either $ux \in \partial G[L_2]$ or $wx \in \partial G[L_2]$. In the first case, let $M = (M' \setminus \{vx\}) \cup \{ux, vw\}$; in the second, let $M = (M' \setminus \{vx\}) \cup \{wx, uw\}$. In both cases, M is a matching of $\partial G[L_2]$ with property (1). □

We are now ready to prove Theorem 14. To find the vertices inducing a large outerplanar graph in G, we delete vertices in L_1 until all vertices in L_2 are "exposed" to the external face. To ensure that the resulting outerplanar graph is sufficiently large, we delete vertices in L_1 that expose 2 vertices in L_2 or otherwise ensure 2 vertices will be included in the outerplanar graph.

Let M be a matching as guaranteed by Lemma 4. We create a list K of triples such that each vertex in L_2 occurs in exactly one triple. For each $u \in L_2$ not covered by M, u has between degree at least 2, and we add $\{u, v, w\}$ to K, where v, w are neighbors of u in L_1. For each edge $xy \in M$, by Observation 15, there exists $z \in L_1$ such that xyz is a face, and we add $\{x, y, z\}$ to K.

We then delete vertices from L_1 as follows:

```
while there exists {u,v,w} ∈ K such that {u,v,w} ∩ L_1 = {v}
        delete v from G and delete all triples containing v from K;
while there exists v ∈ L_1 such that v is in 2 or more distinct triples of K
        delete v from G and delete all triples containing v from K;
while {u,v,w} ∈ K
        delete v ∈ L_1 from G and delete {u,v,w} from K.
```

Note that if $v \in L_1$ is deleted from G, all L_2 vertices in a triple with v are exposed. Therefore, the undeleted vertices induce an outerplanar subgraph of G.

In the first two steps, at least two L_2 vertices were exposed for every deleted L_1 vertex. In the final step, all triples are disjoint, so each deletion of an L_1 vertex exposes one L_2 vertex and ensures that one L_1 vertex will not be deleted; again, 2 vertices are included in the induced outerplanar subgraph for every deleted vertex. This means that the subgraph contains at least two thirds of the vertices of G. This complete the proof of Theorem 14. □

This result is tight, as the disjoint union of multiple octahedrons (see Figure 4) is 2-outerplanar, and its largest induced outerplanar subgraph is on $\frac{2}{3}$ of its vertices. The result is also tight for arbitrarily large connected 2-outerplanar graphs, as the same property holds for graphs constructed by connecting disjoint octahedrons as shown in Figure 5.

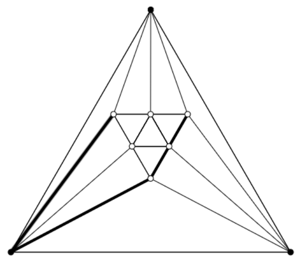

Fig. 4. The outerplanar induced subgraph of this graph found by the algorithm in Theorem 14 is induced by the white vertices. Every induced forest on at least half of the vertices of this graph (an example is shown by the bolded edges) includes vertices not in this outerplanar subgraph.

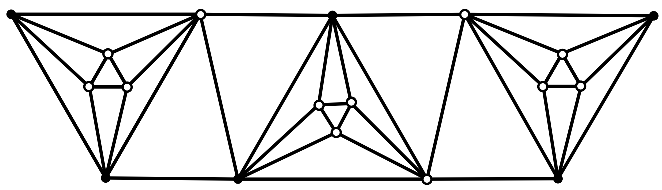

Fig. 5. A 2-outerplanar graph whose largest induced outerplanar subgraph is on $\frac{2}{3}$ of its vertices. The white vertices induce such a subgraph, found by the algorithm in Theorem 14.

Theorem 14 has an immediate corollary for k-outerplanar graphs.

Corollary 2. *Let G be a k-outerplanar graph on n vertices. Then G has an induced $\lceil \frac{k}{2} \rceil$-outerplanar subgraph on at least $\frac{2n}{3}$ vertices.*

Proof. We apply Theorem 14 to pairs of successive layers $\{L_1, L_2\}, \{L_3, L_4\}, \ldots$ in G, finding large induced outerplanar subgraphs $H_i \subseteq G[L_i \cup L_{i-1}]$ for $i = 1, 3, 5, \ldots, k-1$ (if k is even; if k is odd, we end at $i = k-2$). Let $V' := \cup_i V(H_i)$. $G[V']$ is $\lceil \frac{k}{2} \rceil$-outerplanar, as $L_i(G[V']) = V(H_i)$, and $|V'| \geq \frac{2n}{3}$, as $|V(H_i)| \geq \frac{2}{3}|L_i \cup L_{i-1}|$. □

4.1 Future directions

We define an *induced outerplane graph* of a planar graph G is an induced subgraph of G whose embedding inherited from G is an outerplar embedding. We point out that our last result implies an improvement to a graph drawing result of Angelini, Evans, Frati, and Gudmundsson [3]. A simultaneous embedding with fixed edge and without mapping (SEFENoMap) of two planar graphs G_1 and G_2 of the same size n is a planar embedding algorithm that maps any vertex of G_1 into any vertex of G_2 such that: (i) vertices of both graphs are mapped to the same point set in a plane and (ii) every edge that belongs to both G_1 and G_2 must be represented by the same curve in the drawing of two graphs. The OptSEFENoMap problem asks for the maximum $k \leq n$ such that: given any two planar graphs G_1 and G_2 of size n and k, respectively, there exists an induced subgraph G_1' of G_1 such that the planar embedding of G_1' is inherited from G and G_1' and G_2 has a SEFENoMap. The result of Gritzmann et al. [14], implies that k can be as large as the size of any induced outerplane graph of G_1. Angelini, Evans, Frati, and Gudmundssonn (Theorem 1 [3]) showed that any planar graph G of size n has an induced outerplane graph of size at least $\lceil n/2 \rceil$ which implies $k \geq \lceil n/2 \rceil$ by the result of Gritzmann et al. [14]. Our Theorem 14 implies the following corollary, which is an improvement of the result of Angelini, Evans, Frati, and Gudmundssonn.

Corollary 3. *Every n-vertex 2-outerplanar graph and every $\lceil 2n/3 \rceil$-vertex planar graph has a* SEFENoMap*.*

Based on Theorem 14, we conjecture that:

Conjecture 1. Any 3-outerplanar graph on n vertices contains an induced outerplane graph of size at least $\frac{2n}{3}$.

If this conjecture is true, it would, by Hosono's result, imply that the largest induced forest of 3-outerplanar graphs on n vertices has size at least $\frac{4n}{9}$, that is an improvement over Borodin's result. It also improves the result of Angelini, Evans, Frati, and Gudmundssonn for 3-outerplanar graphs. We note that in the proof of Theorem 14, we only need to delete vertices in L_1, and leave L_2 untouched, to get a large induced outerplane graph of 2-outerplanar graphs. For 3-outerplanar graph, one may need to delete vertices in L_3 as shown by Figure 6.

We also believe that that following conjecture, which is also mentioned in in [3], is true:

Conjecture 2. A planar graph on n vertices contains an induced outerplane graph of size at least $\frac{2n}{3}$.

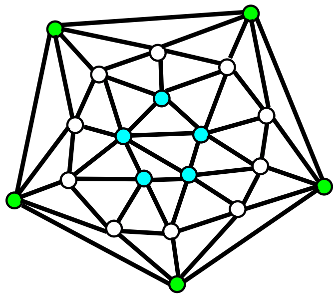

Fig. 6. A 3-outerplanar graph with 20 vertices. Filled vertices are in odd layers and hollow vertices are in even layers. To obtain an induced outerplane graph of size at least 14 ($\lceil 2n/3 \rceil$) vertices, one need to delete at least one vertex in the innermost layer
.

If this conjecture is would imply an improvement of Borodin's result and Angelini, Evans, Frati, and Gudmundssonn' result for general planar graphs.

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