Chromatic Ramsey number of acyclic hypergraphs

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This paper extends to hypergraphs a question discussed in Bialostocki and Gyárfás [4] and Garrison [8]: can one generalize a Ramsey type result from complete host graphs to graphs of sufficiently large chromatic number?

For an \( r \)-uniform tree \( T \) \((r \geq 2)\) we define the \((t,\text{color})\) chromatic Ramsey number \( \chi(T, t) \) as the smallest \( m \) with the following property: if the edges of any \( m \)-chromatic \( r \)-uniform hypergraph are colored with \( t \) colors in any manner, there is a monochromatic copy of \( T \). The presence of a tree is not accidental: \( \chi(H, t) \) can be defined only for an acyclic hypergraph \( H \) since there are hypergraphs with arbitrarily large chromatic number and girth.

We prove that
\[
\left\lceil \frac{R^r(T, t) - 1}{r - 1} \right\rceil + 1 \leq \chi(T, t) \leq |E(T)|^t + 1
\]
where \( R^r(T, t) \) is the \( t \)-color Ramsey number of \( T \). We give better upper bounds for \( \chi(T, t) \) when \( T \) is a matching or a star, proving that for \( r \geq 2 \), \( k \geq 1 \), \( t \geq 1 \), \( \chi(M_r^k, t) \leq (t - 1)(k - 1) + 2k \) and \( \chi(S_r^k, t) \leq t(k - 1) + 2 \) where \( M_r^k \) and \( S_r^k \) are, respectively, the \( r \)-uniform matching and star with \( k \) edges.

The general upper bounds are improved for 3-uniform hypergraphs. We prove that \( \chi(M_r^3, 2) = 2k \), extending a special case of Alon–Frankl–Lovász theorem. We also prove that \( \chi(S_r^3, t) \leq t + 1 \), which is sharp for \( t = 2, 3 \). This is a corollary of our main result which bounds the chromatic number \( \chi(H) \) of 3-uniform hypergraphs by the chromatic number of its 1-intersection graph \( H^{[1]} \), whose vertices represent hyperedges and whose edges represent intersections of hyperedges in exactly one vertex. We prove that \( \chi(H) \leq \chi(H^{[1]}) \) for any 3-uniform hypergraph \( H \) (assuming that \( H^{[1]} \) has at least one edge). The proof uses the list coloring version of Brooks’ theorem. The more general question, whether \( \chi(H) \leq \chi(H^{[1]}) \) holds for every \( r \)-uniform hypergraph \( (r > 3) \) remains open. We could not decide either whether the above lower bound of \( \chi(T, t) \) is sharp for every \( r \)-uniform tree.

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1. Introduction, results

A hypergraph $H = (V, E)$ is a set $V$ of vertices together with a nonempty set $E$ of subsets of $V$, which are called edges. In this paper, we will assume that for each $e \in E$, $|e| \geq 2$. If $|e| = r$ for each $e \in E$, then $H$ is $r$-uniform; a 2-uniform $H$ is a graph. A hypergraph $H$ is acyclic if $H$ contains no cycles (including 2-cycles which are two edges intersecting in at least two vertices). If $H$ is a connected acyclic hypergraph, we say that $H$ is a tree. In particular, a star is a tree in which exactly one vertex is common to every edge. A matching is a hypergraph consisting of pairwise disjoint edges, with every vertex belonging to some edge. We denote by $S^r_k$ and $M^r_k$ the $r$-uniform $k$-edge star and matching, respectively.

For a positive integer $k$, a function $c : V \to \{1, \ldots, k\}$ is called a $k$-coloring of $H$. A coloring $c$ is proper if no edge of $H$ is monochromatic under $c$. The chromatic number of $H$, denoted $\chi(H)$, is the least $m \geq 1$ for which there exists a proper $m$-coloring of $H$ and in this case, we say that $H$ is $m$-chromatic. Given $H = (V, E)$, a partition $(E_1, \ldots, E_t)$ of $E$ into $t$ parts is called a $t$-edge-coloring of $H$. For $r$-uniform hypergraphs $H_1, H_2, \ldots, H_t$, the ($t$-color) Ramsey number $R^r_t(H_1, H_2, \ldots, H_t)$ is the smallest integer $n$ for which the following is true: under any $t$-edge-coloring of the complete $r$-uniform hypergraph $K^n_r$, there is a monochromatic copy of $H_i$ in color $i$ for some $i \in \{1, 2, \ldots, t\}$. When all $H_i = H$ we use the notation $R^r_t(H)$.

Bialostocki and the senior author of this paper extended two well-known results in Ramsey theory from the complete host graph $K_n$ to arbitrary $n$-chromatic graphs [4]. One extends a remark of Erdős and Rado stating that in any $2$-coloring of the edges of a complete graph $K_n$, there is a monochromatic spanning tree. The other is the extension of the result of Cockayne and Lorimer [5] about the $t$-color Ramsey number of matchings. In [8], an acyclic hypergraph $H$ is defined as $t$-good if every $t$-edge coloring of any $K^t_r(H, t)$-chromatic graph contains a monochromatic copy of $H$. Matchings are $t$-good for every $t$ [4] and in [8] it was proved that stars are $t$-good, as well as the path $P_4$ (except possibly for $t = 3$). Additionally, $P_5, P_6, P_7$ are $2$-good. In fact, as remarked in [4], there is no known example of an acyclic $H$ that is not $t$-good.

In this paper, we explore a similar extension of Ramsey theory for hypergraphs, motivating the following definition.

Definition 1. Suppose that $T$ is an acyclic $r$-uniform hypergraph. Let $\chi(T, t)$ be the smallest $m$ with the following property: under any $t$-edge-coloring of any $m$-chromatic $r$-uniform hypergraph, there is a monochromatic copy of $T$.

We call $\chi(T, t)$ the chromatic Ramsey number of $T$. It follows from the existence of hypergraphs of large girth and chromatic number that the chromatic Ramsey number can be defined only for acyclic hypergraphs.

1.1. General trees and acyclic hypergraphs

To see that $\chi(T, t)$ is indeed well defined, we use the following result.

Lemma A ([10,12]). If $H$ is $r$-uniform with $\chi(H) \geq k + 1$, then $H$ contains a copy of any $r$-uniform tree on $k$ edges.

Proposition 2. For any $r$-uniform tree $T$, $\chi(T, t) \leq |E(T)|^t + 1$.

Proof. Fix $t \geq 1$. Let $T$ be an $r$-uniform tree with $k$ edges and let $H = (V, E)$ be a hypergraph with $\chi(H) \geq k^t + 1$. Let $E = E_1 \cup \cdots \cup E_t$ be a $t$-coloring of its edges and set $H_t = (V, E_t)$.

Then $\chi(H_t) = \chi(H_t) \geq k^t + 1$ holds and without loss of generality, $\chi(H_t) = k + 1$. By Lemma A, $H_t$ contains a copy of $T$. $\square$

The simple bound of Proposition 2 can be easily reduced for graphs.

Proposition 3. For any 2-uniform tree $T$ with $k$ edges, $\chi(T, t) \leq 2kt + 1$.

Proof. Let $G$ be a $t$-colored graph with $\chi(G) \geq 2kt + 1$ and $T$ is a tree with $k$ edges. Clearly $G$ has a subgraph of minimum degree $2kt$, with a slight abuse of notation, we keep the name $G$ for it. Then, for some color $i$, $\sum_{v \in V(G)} d_i(v) \geq 2k|V(G)|$, where $d_i(v)$ is the number of edges in color $i$ incident to $v$. Thus the graph $G_i$, whose edges are the edges of $G$ with color $i$, has average degree at least $2k$ therefore $G_i$ has a subgraph of minimum degree $k$ which must contain every tree with $k$ edges. $\square$

Question 4. Let $T$ be an $r$-uniform tree. Is there an upper bound for $\chi(T, t)$ which is linear in both $t, |E(T)|$?

Since any $r$-uniform acyclic hypergraph $T$ may be found in some $r$-uniform tree $T'$, $\chi(T, t)$ is well-defined for any $r$-uniform acyclic hypergraph. Observe the following natural lower bound of $\chi(T, t)$. Let $L(T, t, r) := \left\lceil \frac{R^r_t(T) - 1}{r-1} \right\rceil + 1$.

Proposition 5. For any acyclic $r$-uniform $T$, $L(T, t, r) \leq \chi(T, t)$.

Proof. Let $N := R^r_t(T) - 1$. By the definition of the Ramsey number, there is a $t$-coloring of the edges of $K^n_r$ without a monochromatic $T$. Since $\chi(K^n_r) = \lceil \frac{n}{r-1} \rceil$, the proposition follows. $\square$
The notion of t-good graphs can be naturally extended to hypergraphs using Proposition 5. An acyclic r-uniform hypergraph $T$ is called t-good if every t-edge coloring of any $l(T, t, r)$-chromatic r-uniform hypergraph contains a monochromatic copy of $T$. In other words, $T$ is t-good if $l(T, t, r) = \chi(T, t)$. Note that for $r = 2$, this gives the definition of good graphs. Although it is unlikely that all acyclic hypergraphs are t-good, we have no counterexamples.

For special families of r-uniform acyclic hypergraphs, namely for stars and matchings, we found better upper bounds for $\chi(T, t)$. Surprisingly, most of the bounds attained do not depend on $r$.

1.2. Stars

**Proposition 6.** $\chi(S_k^r, t) \leq t(k - 1) + 2$.

**Proof.** Fix $t, k \geq 1$ and let $p := t(k - 1) + 2$. Suppose that $H$ is r-uniform with $\chi(H) \geq p$ and its edges are t-colored. By Lemma A, $\chi(S_{p-1}^r, 1) \leq p$, so we can find a copy of $S_{p-1}^r$ in $H$. By the pigeonhole principle, $k$ of the edges of $S_{p-1}^r$ have the same color, forming a monochromatic copy of $S_k^r$. □

How good is the estimate of Proposition 6? Notice first that for $t = 1$ it is sharp.

**Proposition 7.** $\chi(S_k^r, 1) = k + 1$.

**Proof.** Consider the complete hypergraph $K = K_{k(r-1)}$. Clearly, $\chi(K) = k$ and $S_k^r$ is not a subgraph of $K$, as its vertex set is too large. □

If $t = 2$, Proposition 6 gives $\chi(S_k^r, 2) \leq 2k$. For $r = 2$ and odd $k$, this is a sharp estimate. For $k = 1$, this is trivial; for $k \geq 3$, the complete graph $K_{2k-1}$ can be partitioned into two $(k - 1)$-regular subgraphs. However, for even $k$, $\chi(S_k^r, 2) = 2k - 1$.

An interesting problem arises when $T = S_k^r$ with $r \geq 3$, when Proposition 6 gives the upper bound $t + 2$. We can decrease this bound by introducing the notion of 1-intersection graphs of a hypergraph.

**Definition 8.** Let $H = (V(H), E(H))$ be a hypergraph. The 1-intersection graph of $H$ is denoted $H^{[1]}$, where $V(H^{[1]}) = E(H)$ and

$$E(H^{[1]}) = \{(e, f) : e, f \in E(H) \text{ and } |e \cap f| = 1\}.$$ 

It is well-known that if $H^{[1]}$ is trivial, i.e., no two edges of $H$ intersect in exactly one vertex, then $H$ is 2-colorable ([14], Exercise 13.33). Note that the stronger statement $\chi(H) \leq \chi(H^{[1]}) + 1$ follows from applying the greedy coloring algorithm in any order of the vertices of $H$. Can we improve this?

**Question 9.** Let $r \geq 3$. Is it true that $\chi(H) \leq \chi(H^{[1]})$ for any $r$-uniform hypergraph $H$, provided $H^{[1]}$ has at least one edge?

Our main result is the positive answer to Question 9 for the 3-uniform case and its corollary.

**Theorem 10.** If $H$ is a 3-uniform hypergraph with at least one edge in $H^{[1]}$ then $\chi(H) \leq \chi(H^{[1]}).

**Corollary 11.** For $t \geq 1$, $\chi(S_3^r, t) \leq t + 1$.

**Proof.** Suppose that we have a $t$-coloring $c$ on the edges of a 3-uniform $H$ with $\chi(H) \geq t + 1$. We claim that there are two edges of $H$ with the same color that intersect in one vertex, defining a monochromatic $S_3^r$. Indeed, by Theorem 10 we have $t + 1 \leq \chi(H) \leq \chi(H^{[1]})$. Then $c$ defines a vertex $t$-coloring on the (at least) $(t + 1)$-chromatic graph $H^{[1]}$, this coloring cannot be proper: there are two vertices of the same color forming an edge in $H^{[1]}$ and proving the claim. □

The case $t = 2$ of Corollary 11 was the initial aim of the research in this paper and it was proved first by Zoltán Füredi [7]. Our proof of Theorem 10 uses his observation (Lemma 16) and the list-coloring version of Brooks’ theorem. Corollary 11 is obviously sharp for $t = 2$; it follows from Proposition 5 that it is also sharp for $t = 3$, because $R^3(S_3^r, 3) = 6$ [3]. It would be interesting to see whether Corollary 11 is true for any $S_3^r$ (in particular for $r = 4, t = 2$) as this is equivalent to the statement that $r$-uniform hypergraphs with bipartite 1-intersection graphs are 2-colorable.

1.3. Matchings

When $T$ is a matching, the bounds of $\chi(T, t)$ relate to the following well-known result of Alon, Frankl, and Lovász (originally conjectured by Erdős).

**Theorem B ([1]).** For $t \geq 2$, $r \geq 1, t \geq 1,$

$$R^t(M_k^r, t) = (t - 1)(k - 1) + kr.$$ 

Note that special cases of Theorem B include $r = 2$ [5], $k = 2$ [13], $t = 2$ [2,9].
We obtain the following linear upper bound for matchings using Theorem B.

**Theorem 12.** For \( r \geq 2, k \geq 1, t \geq 1 \), \( \chi(M^r_k, t) \leq (t - 1)(k - 1) + 2k \). Equality holds for \( r = 2 \).

We tighten this bound, provided \( r \geq 3 \) and \( t = 2 \).

**Theorem 13.** For \( r \geq 3 \) and \( k \geq 1 \), \( \chi(M^r_k, 2) \leq 2k \).

We get two corollaries of Theorem 13 when its upper bound coincides with the lower bound of Proposition 5.

**Corollary 14.** \( \chi(M^3_k, 2) = 2k \).

**Corollary 15.** For \( r \geq 3 \), \( \chi(M^r_k, 2) = 4 \).

Corollary 14 extends Theorem B (for \( r = 3, t = 2 \)) because \( \chi(K^3_{4k-1}) = 2k \). However, Corollary 15 does not extend Theorem B for \( r \geq 4 \). Indeed, for \( r = 4 \), the bound \( \left\lceil \frac{12r+4}{r-1} \right\rceil \) derived from Theorem B is 3.

2. Proofs of main results

2.1. Proof of Theorem 10

In this section, we use the phrase “triple system” for a 3-uniform hypergraph. The word “triple” will take the place of “edge” so that “edge” may be reserved for graphs. Our goal is to construct a proper \( t \)-coloring of \( H \) from a proper \( t \)-coloring of \( H^{[r]} \). Note that a partition of \( E(H) \) into classes \( E_1, E_2, \ldots, E_t \) such that for any \( i, 1 \leq i \leq t \), no two edges of \( E_i \) 1-intersect is precisely a proper \( t \)-coloring of \( H^{[r]} \).

A triple system is connected if for every partition of its vertices into two nonempty parts, there is a triple intersecting both parts. Every triple system can be uniquely decomposed into pairwise vertex-disjoint connected parts, called components. Components with one vertex are called trivial components.

Let \( B_k \) denote the triple system with \( k \) edges intersecting pairwise in the vertices \( \{v, w\} \), called the base of \( B_k \). A \( B \)-component (also, \( B_k \)-component) is a triple system which is isomorphic to \( B_k \) for some \( k \geq 1 \). A \( K \)-component is either three or four distinct triples on four vertices.

**Lemma 16.** Let \( C \) be a nontrivial component in a triple system without 1-intersections. Then \( C \) is either a \( B \)-component or a \( K \)-component.

**Proof.** If \( C \) has at most four vertices then \( 1 \leq |E(C)| \leq 4 \) (where \( E(C) \) is here considered as a set, not a multiset) and by inspection, \( C \) is either \( B_1 \), \( B_2 \), or a \( K \)-component. Assume \( C \) has at least five vertices and select the maximum \( m \) such that \( e_1, e_2, \ldots, e_m \in E(C) \) are distinct triples intersecting in a two-element set, say in \( \{x, y\} \). Clearly, \( m \geq 2 \). Then \( A = \bigcup_{i=1}^{m} e_i \) must cover all vertices of \( C \), as otherwise there is an uncovered vertex \( z \) and a triple \( f \) containing \( z \) and intersecting \( A \), since \( C \) is a component. However, from \( m \geq 2 \) and the intersection condition, \( f \cap A = \{x, y\} \) follows, contradicting the choice of \( m \). Thus \( A = V(C) \) and from \( |V(C)| \geq 5 \) we have \( m \geq 3 \). It is obvious that any triple of \( C \) different from the \( e_i \)'s would intersect some \( e_j \) in one vertex, violating the intersection condition. Thus \( C \) is isomorphic to \( B_m \), concluding the proof. □

A multigraph \( G \) is called a skeleton of a triple system \( H \) if every triple contains at least one edge of \( G \). We may assume that \( V(H) = V(G) \). A matching in a multigraph is a set of pairwise disjoint edges. A factorized complete graph is a complete graph on \( 2m \) vertices whose edge set is partitioned into \( 2m - 1 \) matchings. The following lemma allows us to define a special skeleton of triple systems.

**Lemma 17.** Suppose that \( H \) is a triple system with \( \chi(H^{[r]}) = t \geq 2 \) and \( H_1, H_2, \ldots, H_t \) is a partition of \( H \) into triple systems (each \( H_i \) is considered on vertex set \( V(H) \)) where each \( H_i \) has no 1-intersections. Then there exists a skeleton \( G \) of \( H \) with the following properties:

1. \( E(G) = \bigcup_{i=1}^{t} E_i \) where each \( E_i \) is a matching and a skeleton of \( H_i \).
2. For \( 1 \leq i \leq t \), edges of \( E_i \) are the bases of all \( B \)-components of \( H_i \) and two disjoint vertex pairs from all \( K \)-components of \( H_i \).
3. If \( K^* = K_{i+1} \subset G \) then \( K^* \) is a connected component of \( G \) factorized by the \( M_i \)'s and there is some \( e \in M_1 \cap E(K^*) \) such that \( e \) is from a \( B \)-component of \( H_i \).

**Proof.** From Lemma 16 we can define \( M_i \) by selecting the base edges from every \( B \)-component of \( H_i \) and selecting two disjoint pairs from every \( K \)-component of \( H_i \). The resulting multigraph is clearly a skeleton of \( H \) and satisfies properties 1 and 2. We will select the disjoint pairs from the \( K \)-components so that property 3 also holds. Notice that \( K^* = K_{i+1} \subset G \) must form a connected component in \( G \) because it is a \( t \)-regular subgraph of a graph of maximum degree \( t \). Also, \( K_{i+1} \) is factorized by the \( M_i \)'s because the union of \( t \) matchings can cover at most \( \frac{t(t+1)}{2} = \binom{t+1}{2} \) edges of \( K_{i+1} \), therefore every edge of \( K_{i+1} \) must be
covered exactly once by the $M_i$'s. Thus we have to ensure only that there is $e \in M_1 \cap E(K^*)$ with $e$ from a $B$-component of $H_i$. For convenience, we say that a $K^* = K_{t+1}$ is a bad component if such e does not exist.

Select a skeleton $S$ as described in the previous paragraph such that $p$, the number of bad components, is as small as possible. Suppose that $(x, y) \in M_1$ is a bad component $U$. In other words, $(x, y)$ is in a $K$-component of $H_1$, where $V(K) = \{x, y, u, v\}$ and $(u, v) \in M_1$. Now we replace these two pairs by the pairs $(x, u), (y, v)$ to form a new $M_1$. After this switch, $U$ is no longer a bad component. In fact, either $U$ becomes a new component on the same vertex set (if $(u, v)$ was in $U$) or $U$ melds with another component into a new component. In both cases, no new bad components are created and in the new skeleton there are fewer than $p$ bad components. This contradiction shows that $p = 0$ and proves the lemma.

**Proof of Theorem 10.** Let $H$ be a triple system with $t := \chi(H^{[1]}) \geq 2$ and partition $H$ into $H_1, \ldots, H_t$ so that each $H_i$ is without 1-intersections. Let $G$ be a skeleton of $H$ with the properties ensured by Lemma 17.

Let $G'$ be a connected component of $G$. By Brooks' Theorem, if $G'$ is not the complete graph $K_{t+1}$ or an odd cycle (if $t = 2$), $\chi(G') \leq \Delta(G') \leq t$.

Suppose first that $t$ is even. Now $G' \neq K_{t+1}$ because that would contradict property 3 in Lemma 17: $K_{t+1}$ cannot be factorized into matchings. Also, for $t = 2$, $G'$ cannot be an odd cycle since odd cycles are not the union of two matchings. Thus every connected component of $G$ is at most $t$-chromatic, therefore $\chi(G) \leq t$. Since $G$ is a skeleton of $H$, this implies $\chi(H) \leq t$, concluding the proof for the case when $t$ is even.

Suppose that $t$ is odd, $t \geq 3$. In this case the previous argument does not work when some connected component $G' = K_{t+1} \subset G$. However, from Lemma 17, every $K_{t+1}$-component of $G$ has an edge $(x_i, y_i) \in M_1$ that is the base of a $B$-component in $H_i$. Define the vertex coloring $c$ on $X = \bigcup_{k=1}^{t} V(C_k)$ by $c(x_i) = c(y_i) = 1$ and by coloring all the other vertices of all $C_k$'s with $2, \ldots, t$.

Let $F$ be the subgraph of $G$ spanned by $V(G) \setminus X$ and define

$$Z := \{z \in V(F) : (x_i, y_i, z) \in E(H_1) \text{ for some } 1 \leq i \leq m\}.$$ Fix any $z \in Z$. Then there is a triple $T = (x_i, y_i, z) \in H_1$ in a $B$-component of $H_i$ where $T$ has base $(x_i, y_i)$. If $(z, u) \in M_1$ for some $u \in V(G)$, some triple $S$ containing $(z, u)$ would be in $H_2$. But then $S, T$ would 1-intersect in $z$, contradicting to the definition of $H_1$. Thus $d_G(z) = t - 1$ for all $z \in Z$. Note also that $d_G(u) \leq t$ for all $u \in V(F) \setminus Z$.

We claim that with lists $L(z) := \{2, \ldots, t\}$ for $z \in Z$ and $L(v) := \{1, \ldots, t\}$ for $v \in V(F) \setminus Z$, $F$ is L-choosable. We use the reduction argument present in many coloring proofs (see, for example, the very recent survey paper [6]).

Suppose $F$ is not L-choosable and let $F'$ be a minimal induced subgraph of $F$ which fails to be L-choosable. We may assume that any $z \in V(F') \cap Z$ has $d_F(z) = t - 1$ (otherwise we may L-choose $F' - z$, add $z$ back and properly color it). Likewise we may assume $d_F(v) = t$ for all $v \in V(F') \setminus Z$. By the degree-choosability version of Brooks' theorem (see [11], Lemma 1 or [6], Theorem 11), $F'$ is a Gallai tree: a graph whose blocks are complete graphs or odd cycles.

Let $A$ be an endblock of $F'$. Then $A \neq K_{t+1}$ because all $K_{t+1}$-components of $G$ are in $X$. Since all vertex degrees in $F'$ are $t$ or $t - 1$, $A$ is either an odd cycle (if $t = 3$) or $A$ is a $K_t$. $A$ must contain an edge $e \in M_1$. Otherwise $M_2, \ldots, M_t$ would cover the edges of $A$, a contradiction in either case. By the degree requirements, either $V(A) \cap (V(F) \setminus Z) = \{w\}$ where $w$ is the unique cut point of $A$ or $V(A) \subset Z$. In both cases an endpoint of $e$ must be in $Z$. Then there exists some triple $(x_i, y_i, z) \in H_1$ which 1-intersects with the triple of $H_1$, containing $e$, a contradiction to the definition of $H_1$, proving that $F$ is L-choosable.

Let $c' : V(F) \to \{1, \ldots, t\}$ be an L-coloring of $F$. We extend $c$ from $X$ to $V(H)$ by setting $c(v) := c'(v)$ for all $v \in V(F)$. Observe that $c$ properly colors all edges of $G$ except for the edges of the form $(x_i, y_i)$ which are monochromatic in color 1. Since $G$ is a skeleton, every triple of $H$ is properly colored except possibly the triples in the from $(x_i, y_i, x)$.

We claim that $c(x) \neq 1$. Suppose to the contrary that $c(x) = 1$. If $x \in X$ then $x \in \{x_i, y_i\}$ for some $j \neq i$, but this is impossible because the bases $(x_i, y_i), (x_j, y_j)$ are from different $B$-components of $H_1$. If $x \not\in X$ then $x \in Z$ from the definition of $Z$. However, $1 \not\in L(x)$ for $x \in Z$ and this proves the claim.

Therefore $c$ is a proper $t$-coloring of $H$ and this completes the proof.

2.2. Proof of Theorem 12

Let $H = (V, E)$ be an r-uniform hypergraph with $\chi(H) \geq p$ where

$$p = (t - 1)(k - 1) + 2k.$$ Consider any $t$-edge coloring $\{E_1, \ldots, E_t\}$ of $H$ and any proper coloring $c$ of $H$ obtained by the greedy algorithm (under any ordering of its vertices). Clearly $c$ uses at least $p$ colors and for any $1 \leq i < j \leq p$ there is an edge $e_{ij}$ in $H$ whose vertices are colored with color $i$ apart from a single vertex which is colored with $j$. Let $\{F_1, \ldots, F_t\}$ be a $t$-edge-coloring of $K_2^2$ defined so that $F_i := \{i, j\} : 1 \leq i < j \leq p, e_{ij} \in E_i$ for each $s$. $1 \leq s \leq t$. From the definition of $p$, Theorem B (in fact the Cockayne–Lovász Theorem suffices) implies that there is a monochromatic $M_k^2$ in $K_2^2$. Observe that

$$\{e_{ij} : (i, j) \in M_k^2\}.$$
is a set of $k$ pairwise disjoint edges in $H$ in the same partition class of $\{E_1, \ldots, E_r\}$. This completes the proof that $\chi(M_k^r, t) \leq (t - 1)(k - 1) + 2k$. The lower bound $R^2(M_k^2, t) \leq \chi(M_k^2, t)$ implies equality in the $r = 2$ case. □

2.3. Proof of Theorem 13

We fix $r \geq 3$ and proceed by induction on $k$. Suppose $k = 1$ and let $H$ be some $r$-uniform hypergraph with $\chi(H) \geq 2$. Then any 2-edge-coloring of $H$ contains a single monochromatic edge since $H$ has at least one edge. Now suppose the theorem is true for $k - 1 \geq 1$ and let $H = (V, E)$ be $r$-uniform with $\chi(H) \geq 2k$. Without loss of generality, $H$ is connected. Fix some 2-edge-coloring $\{E_1, E_2\}$ of $H$, calling the edges of $E_1$ “red” and the edges of $E_2$ “blue”. If $E_1$ or $E_2$ is empty, then Theorem 12 with $t = 1$ implies the desired bound.

So we may assume otherwise, and there exist edges $e, f \in E$ with $e$ red and $f$ blue. Let $s := |e \cap f|$ and $A := e \cup f$. If $H[A]$ is 2-colorable, then $\chi(H - A) \geq \chi(H) - 2 \geq 2(k - 1)$ so by induction we find a monochromatic $M_{k-1}^r$ matching in $H - A$. Without loss of generality, $M_{k-1}^r$ is red and $M_{k-1}^r + e$ is a red $M_k^r$ in $H$.

If $s > 1$, then $|A| = 2r - s \leq 2r - 2$ thus $H[A]$ is certainly 2-colorable and the induction works. If $s = 1$ and $H[A]$ is not 2-colorable then $H[A]$ is $K_{r+1}^r$. Writing $e = \{w, u_1, \ldots, u_{r-1}\}$ and $f = \{w, v_1, \ldots, v_{r-1}\}$, the edge $g = \{w\} \cup \{u_1, u_3, \ldots\} \cup \{v_2, v_4, \ldots\} \in E(H)$. Without loss of generality, $g$ is red and $|g \cap f| = 1 + \lceil (r - 1)/2 \rceil \leq 2$ since $r \geq 3$. So the previous case applies to the red edge $g$ and blue edge $f$. Finally, if $s = 0$ and $H[A]$ is not 2-colorable there must be $g \in H[A]$ that intersects both $e$ and $f$. Then either $e, g$ or $f, g$ is a pair of edges of different colors that intersect, and a previous case can be applied again. □

References