Calculating Perihelion Precession Using The Multiple Scales Method

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0.1 Introduction

While Kepler's First Law famously states that a planet travels in an ellipse with the sun at one focus, it only holds in a system with two spherical bodies under the operation of Newtonian mechanics. In more realistic systems, the various attributes of an orbit—the length of the semi-major axis, the eccentricity, the inclination, the perihelion, and the nodes—vary in time. Since the equations describing these systems are generally impossible to solve exactly, perturbation theory comes into play, but the lack of consistancy among perturbative techniques makes it difficult to ascertain the correctness of the results. By using the *multiple scales* technique, it is possible to quickly and reliably calculate the desired quantities.

0.2 Two-body Newtonian system

The Lagrangian for a two-body system, where both bodies are rigid spheres, is given by

$$L = \frac{1}{2}m_1\left(\dot{x}_{11}^2 + \dot{x}_{12}^2 + \dot{x}_{13}^2\right) + \frac{1}{2}m_2\left(\dot{x}_{21}^2 + \dot{x}_{22}^2 + \dot{x}_{23}^2\right)$$
$$+ \frac{m_1 m_2}{\sqrt{(x_{11} - x_{21})^2 + (x_{12} - x_{22})^2 + (x_{13} - x_{23})^2}}$$

where x_{1i} and x_{2j} are the Cartesian coordinates of the centers of the two bodies, and m_1 and m_2 are their masses. To solve this system, introduce new coordinates by

$$MX_i = m_1 x_{1i} + m_2 x_{2i}, x_i = x_{1i} - x_{2i}$$

$$\Rightarrow x_{1i} = X_i + \frac{m_2}{M} x_i, x_{2i} = X_i - \frac{m_1}{M} x_i$$

where $M = m_1 + m_2$ is the total mass. Then the Lagrangian becomes

$$L = \frac{1}{2} M \left(\dot{X}_1^2 + \dot{X}_2^2 + \dot{X}_3^2 \right) + \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right) + \frac{mM}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

where $m = \frac{m_1 m_2}{M}$ is the reduced mass. The potential does not depend on the center-of-mass coordinates X_i, so the latter can be taken to be fixed at the origin. Introducing spherical coordinates, the Lagrangian becomes

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) + \frac{mM}{r}$$

Minimizing the action, $\int L \, dt$, yields the following system of equations:

$$\frac{d}{dt} (m\dot{r}) = mr \left(\dot{\theta}^2 + \sin^2 \theta \, \dot{\phi}^2 \right) - \frac{mM}{r^2}$$

$$\frac{d}{dt} \left(mr^2 \dot{\theta} \right) = mr^2 \sin\theta \cos\theta \, \dot{\phi}^2$$

$$\frac{d}{dt} \left(mr^2 \sin^2 \theta \, \dot{\phi} \right) = 0$$

Dividing through by the constant mass, m, taking the planar solution $\theta = \frac{\pi}{2}, \dot{\theta} = 0$, and setting the constant $h = r^2 \dot{\phi}$, the r equation becomes

$$\ddot{r} = \frac{h^2}{r^3} - \frac{M}{r^2}$$

Letting $u = \frac{1}{r}$, by the chain rule,

$$\frac{du}{d\phi} = -\frac{1}{r^2} \frac{dr}{d\phi} = -\frac{\dot{r}}{r^2 \dot{\phi}} = -\frac{\dot{r}}{h}$$

and

$$\frac{d^2 u}{d\phi^2} = \frac{1}{\dot{\phi}} \frac{d}{dt} \left(\frac{du}{d\phi} \right) = -\frac{\ddot{r}}{u^2 h^2}$$

Substituting this into the r equation leads to

$$\frac{d^2u}{d\phi^2} + u - \frac{M}{h^2} = 0$$

which can be readily solved, giving

$$u = \frac{M}{h^2}\left(1 + e\cos\left(\phi - \phi_0\right)\right)$$

This is the equation of an ellipse with eccentricity e and with semi-major axis length

$$a = \frac{1}{2}\left(\frac{h^2}{M\left(1-e\right)} + \frac{h^2}{M\left(1+e\right)}\right) \Rightarrow \frac{h^2}{M} = a^2\left(1 - e^2\right)$$

Substituting this into the equation $h = r^2\dot{\phi}$ gives the period of an orbit:

$$T = \int_0^{2\pi} \frac{d\phi}{hu^2} = \frac{h^3}{M^2}\int_0^{2\pi} \frac{d\phi}{\left(1 + e\cos\phi\right)^2}$$

$$\Rightarrow T = 2\frac{h^3}{e^2M^2}\int_{-1}^{1} \frac{dv}{\sqrt{1-v^2}\left(v - \frac{1}{e}\right)^2}$$

This integral can be done using contour methods,

$$T = \frac{h^3}{e^2M^2}\int_\Gamma \frac{dz}{\sqrt{1-z^2}\left(z - \frac{1}{e}\right)^2} = 2\pi i\frac{h^3}{e^2M^2}\frac{d}{dz}\left(\frac{1}{\sqrt{1-z^2}}\right)\Bigg|_{z=-\frac{1}{e}}$$

where Γ is a clockwise contour around the branch cut from -1 to 1, resulting in

$$T = \frac{2\pi h^3}{M^2\left(1-e^2\right)^{\frac{3}{2}}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M}} \Rightarrow M = \frac{\left(2\pi\right)^2 a^3}{T^2}$$

0.3 Precession due to other planets

In systems with more than two bodies, or where the bodies are non-spherical, the perihelion of the orbit will not generally remain fixed, but will rather precess. The calculation can be simplified by ignoring short term oscillations or possible resonant-motion effects through replacing the perturbing body by a mass distribution along its orbit. [15] Consider the case of a planet in a circular orbit with radius a_p and mass m_p. On the plane determined by the circle, at a point q of radius r from the center the potential due to the mass distribution is given by

$$V_p = -\frac{m_p}{2\pi a_p}\int_0^{2\pi}\frac{a_p d\phi}{l} = -\frac{m_p}{2\pi a_p}\int_0^{2\pi}\frac{d\alpha}{\cos\left(\alpha-\phi\right)}$$

where α is the angle to the mass element from the point q, ϕ is the angle to the mass element from the origin, and $l = l(\alpha)$ is the distance to the mass element from q. Using $a_p \cos\phi = l\cos\alpha + r$ and $a_p \sin\phi = l\sin\alpha$, the expression for V_p becomes

$$V_p = -\frac{m_p}{2\pi a_p}\int_0^{2\pi}\frac{d\alpha}{\cos\alpha\cos\phi+\sin\alpha\sin\phi} = -\frac{m_p}{2\pi}\int_0^{2\pi}\frac{d\alpha}{l+r\cos\alpha}$$

Using the Law of Cosines,

$$a_p^2 = r^2 + l^2 - 2rl\cos\left(\pi-\alpha\right) \Rightarrow l^2 + 2r\cos\alpha\, l + r^2 - a_p^2 = 0$$

$$\Rightarrow l = -r\cos\alpha + \sqrt{a_p^2 - r^2\sin^2\alpha}$$

$$\Rightarrow V_p = -\frac{m_p}{2\pi a_p}\int_0^{2\pi}\frac{d\alpha}{\sqrt{1-\frac{r^2}{a_p^2}\sin^2\alpha}} = -\frac{2m_p}{\pi a_p}\mathbf{K}\left(\frac{r}{a_p}\right)$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind. (Note this disagrees with Price and Rush, whose derivation is erroneous, but agrees with Stewart [19], which is given without derivation.) Adding this potential to that from the sun, the effective Lagrangian becomes

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + \frac{mM}{r} + \frac{2mm_p}{\pi a_p}\mathbf{K}\left(\frac{r}{a_p}\right)$$

Repeating the analysis for the two-body case gives rise to the equation

$$\frac{d^2u}{d\phi^2} + u - \frac{M}{h^2} + \frac{2m_p}{\pi h^2\left(ua_p\right)^2}F\left(\frac{1}{ua_p}\right) = 0$$

where $F\left(k\right) = \frac{d}{dk}\mathbf{K}$. Letting $u = \frac{M}{h^2}v$ yields

$$\frac{d^2v}{d\phi^2} + v - 1 + \frac{2\epsilon}{\pi\sqrt{\delta}v^2}F\left(\frac{\sqrt{\delta}}{v}\right) = 0$$

where, assuming $a \ll a_p$, both $\delta = \left(\frac{h^2}{Ma_p}\right)^2 \approx \left(\frac{a\left(1-e^2\right)}{a_p}\right)^2$ and $\epsilon = \frac{m_p}{M}\delta$ are small quantities. Employing the technique of multiple scales, let

$$v = \sum_{i,j=0}^{\infty} v_{ij}\epsilon^i\delta^j$$

and $\Phi_{ij} = \epsilon^i\delta^j\phi$, so by the chain rule

$$\frac{d}{d\phi} = \sum_{i,j=0}^{\infty} \frac{d\Phi_{ij}}{d\phi}\frac{\partial}{\partial\Phi_{ij}} = \sum_{i,j=0}^{\infty} \epsilon^i\delta^j \frac{\partial}{\partial\Phi_{ij}}$$

Then the equation becomes:

$$\left(\frac{\partial}{\partial\Phi_{00}} + \epsilon\frac{\partial}{\partial\Phi_{10}} + \delta\frac{\partial}{\partial\Phi_{01}} + \cdots\right)^2 (v_{00} + \epsilon v_{10} + \delta v_{01} + \cdots)$$
$$+ (v_{00} + \epsilon v_{10} + \delta v_{01} + \cdots) - 1$$
$$+\frac{\epsilon}{v^3}\left\{\left(\frac{1}{2}\right)^2 2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 4\frac{\delta}{v^2} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 6\frac{\delta^2}{v^4} + \cdots\right\} = 0$$

The order $\epsilon^0\delta^0$ equation is:

$$\frac{\partial^2 v_{00}}{\partial\Phi_{00}^2} + v_{00} - 1 = 0 \Rightarrow v_{00} = 1 + Be^{i\Phi_{00}} + \overline{B}e^{-i\Phi_{00}}$$

where B is a complex-valued function independent of Φ_{00}. The order $\epsilon^1\delta^0$ equation is:

$$\frac{\partial^2 v_{10}}{\partial\Phi_{00}^2} + v_{10} = -2\frac{\partial^2 v_{00}}{\partial\Phi_{10}\partial\Phi_{00}} - \frac{1}{2v_{00}^3}$$

To avoid the occurrence of a secular term, the resonant terms proportional to $e^{i\Phi_{00}}$ and $e^{-i\Phi_{00}}$ need to cancel. One way to calculate the appropriate Fourier coefficient of v_{00}^{-3} is to use the binomial series:

$$\left(1 + Be^{i\Phi_{00}} + \overline{B}e^{-i\Phi_{00}}\right)^{-3} = \sum_{l=0}^{\infty} (-1)^l \frac{(l+2)!}{2!l!}\left(Be^{i\Phi_{00}} + \overline{B}e^{-i\Phi_{00}}\right)^l$$

Only the odd values of l contribute to the Fourier coefficient of $e^{i\Phi_{00}}$, which is given by

$$-B\sum_{j=0}^{\infty} \frac{(2j+3)!\,(2j+1)!}{2!\,(2j+1)!j!\,(j+1)!}\left(B\overline{B}\right)^j = -3B\sum_{j=0}^{\infty} \frac{\Gamma\left(j+\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2}\right)j!}\left(4B\overline{B}\right)^j$$

$$= -3B\left(1 - 4B\overline{B}\right)^{-\frac{5}{2}}$$

Another way to get the same result is to use the orthogonality of the complex exponentials and contour integration:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\Phi_{00}}\left(1 + Be^{i\Phi_{00}} + \overline{B}e^{-i\Phi_{00}}\right)^{-3} d\Phi_{00}$$

$$= \frac{1}{2\pi i}\int_{|z|=1} \frac{dz}{z^2\left(1 + Bz + \overline{B}\frac{1}{z}\right)^3}$$

$$\frac{1}{2\pi i B^3}\int_{|z|=1} \frac{z\,dz}{\left(z - \frac{-1+\sqrt{1-4B\overline{B}}}{2B}\right)^3\left(z - \frac{-1-\sqrt{1-4B\overline{B}}}{2B}\right)^3}$$

$$= \frac{1}{2!B^3}\left(\frac{d}{dz}\right)^2\left[\frac{z}{\left(z - \frac{-1-\sqrt{1-4B\overline{B}}}{2B}\right)^3}\right]\Bigg|_{z=\frac{-1+\sqrt{1-4B\overline{B}}}{2B}}$$

$$= -3B\left(1 - 4B\overline{B}\right)^{-\frac{5}{2}}$$

The requirement that the resonant terms cancel becomes:

$$-2i\frac{\partial B}{\partial \Phi_{10}} + \frac{3}{2}B\left(1 - 4B\overline{B}\right)^{-\frac{5}{2}} = 0$$

and its complex conjugate

$$2i\frac{\partial \overline{B}}{\partial \Phi_{10}} + \frac{3}{2}\overline{B}\left(1 - 4B\overline{B}\right)^{-\frac{5}{2}} = 0$$

Writing $B = be^{i\beta}$, and by setting the real and imaginary parts equal to zero these become

$$\frac{\partial b}{\partial \Phi_{10}} = 0$$

and

$$2b\frac{\partial \beta}{\partial \Phi_{10}} + \frac{3}{2}b\left(1 - 4b^2\right)^{-\frac{5}{2}} = 0$$

so b is independent of Φ_{10} and $\beta = -\frac{3}{4}\left(1 - 4b^2\right)^{-\frac{5}{2}}\Phi_{10}+$ some function independent of both Φ_{00} and Φ_{10}.

Concluding, to this order, it is possible to identify b with $\frac{1}{2}e$, so the expression for v is:

$$v = 1 + e\cos\left(\left(1 - \frac{3}{4}\epsilon\left(1-e^2\right)^{-\frac{5}{2}}\right)(\phi - \phi_0)\right)$$

and the angle of precession over the course of one orbit is:

$$2\pi \cdot \frac{3}{4}\epsilon\left(1-e^2\right)^{-\frac{5}{2}} = \frac{3\pi}{2}\left(\frac{m_p}{M}\right)\left(\frac{a}{a_p}\right)^2 \frac{1}{\sqrt{1-e^2}}$$

A more complete analysis, taking into account the eccentricity and inclination of the perturbing body, yields a similar result in the case these quantities are small, as they are in our solar system. [19] A similar calculation can also be done for a perturbing planet with $a_p < a$ or for an oblate central body. [8] [20]

When Urbain J. J. Le Verrier first calculated these perturbations for the orbit of Mercury in 1843, a 16 second error in the timing of its 1845 transit caused him to lose confidence in his result. Returning to the problem in 1859 after his success in the prediction of the location of Neptune (in which luck played a major part since Le Verrier had assumed Bode's Law would hold for the perturber of Uranus' orbit, but Neptune is the one planet for which it does not hold), he was able to show a definite discrepancy between the predictions of Newtonian theory and the motion of Mercury, using this to predict an intra-Mercurial planet. [4] [10] The search for this hypothetical planet, named Vulcan, became a major controversy in late 19th century astronomy, with many claimed sightings in that era still dominated by naked-eye observation using small refracting telescopes. [1] [12] [7] The advent of photography proved the nonexistence of such a body, causing Hugo von Seeliger to propose the zodiacal light particles collectively as the perturbing mass in 1906, a suggestion that became widely accepted at that time. [5] [2] There was great interest in the oblateness of the sun in the 1970's due to the possibilty of fast internal rotation, [18] [3] [9] a hypothesis which has been disproved with improved solar observations and modelling. [17] [21] More recently, the influence of dark matter on orbits has been studied and found insignificant. [14]

0.4 Hall's modified Newtonian approach

In 1894, Asaph Hall, the discoverer of Mars' moons, made the suggestion that altering the exponent of two in the gravitational force law could account for the discrepancy in the calculated position of Mercury. [4] The resulting Lagrangian can be written

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + \frac{mM}{(1+\delta)\,r^{1+\delta}}$$

Repeating the analysis as in the two-body Newtonian case yields the equation

$$\frac{d^2u}{d\phi^2} + u - \frac{M}{h^2}u^\delta = 0$$

which, upon setting $u = \frac{M}{h^2}v$, becomes

$$\frac{d^2v}{d\phi^2} + v - \left(\frac{M}{h^2}\right)^\delta v^\delta = 0$$

Following the multiple scales approach, let

$$v = \sum_{j=0}^{\infty} v_j \delta^j$$

and $\Phi_j = \delta^j \phi$, so by the chain rule

$$\frac{d}{d\phi} = \sum_{j=0}^{\infty} \frac{d\Phi_j}{d\phi}\frac{\partial}{\partial \Phi_j} = \sum_{j=0}^{\infty} \delta^j \frac{\partial}{\partial \Phi_j}$$

Then the equation becomes:

$$\left(\frac{\partial}{\partial \Phi_0} + \delta\frac{\partial}{\partial \Phi_1} + \cdots\right)^2 (v_0 + \delta v_1 + \cdots) + (v_0 + \delta v_1 + \cdots)$$

$$-1 - \delta \log\frac{M}{h^2} - \delta \log v - \frac{\delta^2}{2!}\left(\log\frac{Mv}{h^2}\right)^2 - \frac{\delta^3}{3!}\left(\log\frac{Mv}{h^2}\right)^3 - \cdots = 0$$

The order δ^0 equation is:

$$\frac{\partial^2 v_0}{\partial \Phi_0^2} + v_0 - 1 = 0 \Rightarrow v_0 = 1 + Be^{i\Phi_0} + \overline{B}e^{-i\Phi_0}$$

where B is a complex-valued function independent of Φ_0. The order δ^1 equation is:

$$\frac{\partial^2 v_1}{\partial \Phi_0^2} + v_1 = -2\frac{\partial^2 v_0}{\partial \Phi_1 \partial \Phi_0} + \log\frac{M}{h^2} + \log v_0$$

To avoid the occurrence of a secular term, the resonant terms proportional to $e^{i\Phi_0}$ and $e^{-i\Phi_0}$ need to cancel. One way to calculate the appropriate Fourier coefficient of $\log v_0$ is to use the properties of the log function:

$$\log\left(1 + Be^{i\Phi_0} + \overline{B}e^{-i\Phi_0}\right)$$

$$= \log\left[\frac{\left(1+\frac{1-\sqrt{1-4B\overline{B}}}{2B\overline{B}}Be^{i\Phi_0}\right)\left(1+\frac{1-\sqrt{1-4B\overline{B}}}{2B\overline{B}}\overline{B}e^{-i\Phi_0}\right)}{\frac{1-\sqrt{1-4B\overline{B}}}{2B\overline{B}}}\right]$$

$$= \log\left(1+\frac{1-\sqrt{1-4B\overline{B}}}{2\overline{B}}e^{i\Phi_0}\right) + \log\left(1+\frac{1-\sqrt{1-4B\overline{B}}}{2B}\, e^{-i\Phi_0}\right)$$

$$- \log\frac{1-\sqrt{1-4B\overline{B}}}{2B\overline{B}}$$

where the minus sign is chosen for the square roots to avoid singularities as $B \to 0$. Using the series expansion for $\log(1+z)$ the Fourier coefficient of the $e^{i\Phi_0}$ term is

$$\frac{1-\sqrt{1-4B\overline{B}}}{2\overline{B}}$$

The requirement that the resonant terms cancel becomes:

$$-2i\frac{\partial B}{\partial \Phi_1} + \frac{1-\sqrt{1-4B\overline{B}}}{2\overline{B}} = 0$$

and its complex conjugate

$$2i\frac{\partial \overline{B}}{\partial \Phi_1} + \frac{1-\sqrt{1-4B\overline{B}}}{2B} = 0$$

Writing $B = be^{i\beta}$, and by setting the real and imaginary parts equal to zero these become

$$\frac{\partial b}{\partial \Phi_1} = 0$$

and

$$2b\frac{\partial \beta}{\partial \Phi_1} + \frac{1-\sqrt{1-4b^2}}{2b} = 0$$

so b is independent of Φ_1 and $\beta = -\frac{1-\sqrt{1-4b^2}}{4b^2}\Phi_1 +$ some function independent of both Φ_0 and Φ_1.

Concluding, to this order, it is possible to identify b with $\frac{1}{2}e$, so the expression for v is:

$$v = 1 + e\cos\left(\left(1-\delta\frac{1-\sqrt{1-e^2}}{e^2}\right)(\phi-\phi_0)\right)$$

and the angle of precession over the course of one orbit is:

$$2\pi \cdot \delta \frac{1-\sqrt{1-e^2}}{e^2}$$

For small values of the eccentricity, this approaches a constant, $\pi\delta$, which can be adjusted to match the discrepancy in the orbit of Mercury, but would create new discrepancies for other bodies. [11]

0.5 Cheon's extension to Newton's laws

In 1979, Ki Jae Cheon proposed an additional law of nature to account for the precession of Mercury's perihelion within a Newtonian framework [6]

$$m + \frac{3}{2}V = constant \Leftrightarrow \dot{m} = -\frac{3}{2}\frac{dV}{dt}$$

Since m is no longer a constant, it cannot simply be divided out of the equations of motion, which for the two-body case of an orbit in the $\theta = \frac{\pi}{2}$ plane become:

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\phi}^2 - \frac{mM}{r^2}$$

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0$$

with an extra equation

$$\dot{m} = -\frac{3mM\dot{r}}{2r^2} \Rightarrow m = m_0 e^{\frac{3M}{2r}}$$

Now $h = r^2\dot{\phi}$ is no longer constant, but the product mh is, so

$$\dot{m}h + m\dot{h} = 0 \Rightarrow \dot{m} = -\frac{m\dot{h}}{h}$$

Substituting these into the r equation

$$m\ddot{r} - \frac{m\dot{h}\dot{r}}{h} = \frac{mh^2}{r^3} - \frac{mM}{r^2}$$

Dividing through by m, introducing $u = \frac{1}{r}$ as before, and using the chain rule

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} = -\frac{\dot{r}}{r^2\dot{\phi}} = -\frac{\dot{r}}{h}$$

and

$$\frac{d^2u}{d\phi^2} = \frac{1}{\dot{\phi}}\frac{d}{dt}\left(\frac{du}{d\phi}\right) = -\frac{\ddot{r}}{u^2h^2} + \frac{\dot{h}\dot{r}}{h^3u^2}$$

the r equation becomes

$$\frac{d^2u}{d\phi^2} + u - \frac{M}{h^2} = 0$$

where

$$h = h_0 e^{\frac{-3Mu}{2}}$$

Setting $u = \frac{M}{h_0^2}v$, this becomes

$$\frac{d^2v}{d\phi^2} + v - e^{\epsilon v} = 0$$

where

$$\epsilon = \frac{3M^2}{h_0^2} \approx \frac{3}{(1-e^2)}\left(\frac{2\pi a}{T}\right)^2$$

is small if the planet's speed is small compared to the speed of light. Expanding the exponential function in a series, it is immediately apparent that to first order in ϵ the solution is

$$v = 1 + e\cos\left((1-\epsilon)\left(\phi - \phi_0\right)\right)$$

and the angle of precession over the course of one orbit is:

$$2\pi\epsilon = \frac{6\pi}{(1-e^2)}\left(\frac{2\pi a}{T}\right)^2$$

which matches the result due to general relativity.

The problem with Cheon's approach is that the coefficient of $\frac{3}{2}$ in the mass law was chosen specifically to give the correct result for the perihelion precession. Applying it to a situation of a particle traveling along a line, say the x-axis, with constant force, k, and position, velocity, and mass at time $t = 0$ respectively given by $x = 0$, v_o, and m_0, the equations of motion become:

$$\dot{m} = \frac{3}{2}k\dot{x} \Rightarrow m = m_0 + \frac{3}{2}kx$$

$$\frac{d}{dt}\left(m\dot{x}\right) = k \Rightarrow m_0x + \frac{3}{4}kx^2 = \frac{1}{2}kt^2 + m_0v_0t$$

The limiting velocity as $t \to \infty$ is $\sqrt{\frac{2}{3}} \neq 1$, which would be expected from special relativity. This is also the maximum initial velocity in the negative x-direction permitted to avoid having the mass go to zero along the trajectory, again contradicting special relativity.

0.6 General relativistic effects

Starting with the Shwarzschild metric, $\mathbf{g}$, modified to include the cosmological constant, Λ, [16]

$$\mathbf{g} = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\mathbf{dt}\otimes\mathbf{dt} + \frac{\mathbf{dx}\otimes\mathbf{dx}}{1-\frac{2M}{r}-\frac{\Lambda r^2}{3}}$$

$$+r^2\mathbf{d\boldsymbol{\theta}}\otimes\mathbf{d\boldsymbol{\theta}} + r^2\sin^2\theta\,\mathbf{d\boldsymbol{\phi}}\otimes\mathbf{d\boldsymbol{\phi}}$$

the Christoffel symbols,

$$\Gamma^{\lambda}_{\mu\,\nu}\frac{\partial}{\partial\lambda} = \nabla_{\frac{\partial}{\partial\mu}}\frac{\partial}{\partial\nu}$$

with $\mu,\nu,\lambda \in \{t,r,\theta,\phi)$ are

$$\Gamma^{t}_{t\,r} = \frac{1}{2}g^{tt}\frac{\partial}{\partial r}g_{tt} = \frac{\frac{M}{r^2}-\frac{\Lambda r}{3}}{1-\frac{2M}{r}-\frac{\Lambda r^2}{3}}$$

$$\Gamma^{r}_{t\,t} = -\frac{1}{2}g^{rr}\frac{\partial}{\partial r}g_{tt} = \left(\frac{M}{r^2}-\frac{\Lambda r}{3}\right)\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)$$

$$\Gamma^{r}_{r\,r} = \frac{1}{2}g^{rr}\frac{\partial}{\partial r}g_{rr} = -\frac{\frac{M}{r^2}-\frac{\Lambda r}{3}}{1-\frac{2M}{r}-\frac{\Lambda r^2}{3}}$$

$$\Gamma^{r}_{\theta\,\theta} = -\frac{1}{2}g^{rr}\frac{\partial}{\partial r}g_{\theta\theta} = -r\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)$$

$$\Gamma^{r}_{\phi\,\phi} = -\frac{1}{2}g^{rr}\frac{\partial}{\partial r}g_{\phi\phi} = -r\sin^2\theta\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)$$

$$\Gamma^{\theta}_{r\,\theta} = \frac{1}{2}g^{\theta\theta}\frac{\partial}{\partial r}g_{\theta\theta} = \frac{1}{r}$$

$$\Gamma^{\theta}_{\phi\,\phi} = -\frac{1}{2}g^{\theta\theta}\frac{\partial}{\partial\theta}g_{\phi\phi} = -\sin\theta\,\cos\theta$$

$$\Gamma^{\phi}_{r\,\phi} = \frac{1}{2}g^{\phi\phi}\frac{\partial}{\partial r}g_{\phi\phi} = \frac{1}{r}$$

$$\Gamma^{\phi}_{\theta\,\phi} = \frac{1}{2}g^{\phi\phi}\frac{\partial}{\partial\theta}g_{\phi\phi} = \cot\theta$$

with all others zero. The Riemann tensor is defined as

$$R^{\lambda}_{\kappa\mu\nu}\frac{\partial}{\partial\lambda} = \mathbf{R}\left(\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}\right)\frac{\partial}{\partial\kappa} = \nabla_{\frac{\partial}{\partial\mu}}\nabla_{\frac{\partial}{\partial\nu}}\frac{\partial}{\partial\kappa} - \nabla_{\frac{\partial}{\partial\nu}}\nabla_{\frac{\partial}{\partial\mu}}\frac{\partial}{\partial\kappa} - \nabla_{[\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}]}\frac{\partial}{\partial\kappa}$$

and the Ricci tensor is its contraction

$$\mathbf{Ricci} = \sum_{\mu} \mathbf{d}\boldsymbol{\mu} \left(\mathbf{R} \left(\frac{\partial}{\partial \mu}, \cdot \right) (\cdot) \right)$$

Calculating the relevant components of the Riemann tensor,

$$R^r_{trt} = \frac{\partial}{\partial r}\Gamma^{\,r}_{t\,t} + \Gamma^{\,r}_{t\,t}\Gamma^{\,r}_{r\,r} - \Gamma^{\,r}_{t\,r}\Gamma^{\,r}_{t\,t} = -\left(\frac{2M}{r^3} + \frac{\Lambda}{3}\right)\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)$$

$$R^\theta_{t\theta t} = \Gamma^{\,r}_{t\,t}\Gamma^{\,\theta}_{r\,\theta} = \left(\frac{M}{r^3} - \frac{\Lambda}{3}\right)\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)$$

$$R^\phi_{t\phi t} = \Gamma^{\,r}_{t\,t}\Gamma^{\,\phi}_{r\,\phi} = \left(\frac{M}{r^3} - \frac{\Lambda}{3}\right)\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)$$

$$\Rightarrow R_{tt} = -\Lambda\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)$$

$$R^t_{rtr} = -\frac{\partial}{\partial r}\Gamma^{\,t}_{t\,r} + \Gamma^{\,r}_{r\,r}\Gamma^{\,t}_{t\,r} - \Gamma^{\,t}_{t\,r}\Gamma^{\,t}_{t\,r} = \frac{\frac{2M}{r^3} + \frac{\Lambda}{3}}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}$$

$$R^\theta_{r\theta r} = -\frac{\partial}{\partial r}\Gamma^{\,\theta}_{r\,\theta} + \Gamma^{\,r}_{r\,r}\Gamma^{\,\theta}_{r\,\theta} - \Gamma^{\,\theta}_{r\,\theta}\Gamma^{\,\theta}_{r\,\theta} = \frac{-\frac{M}{r^3} + \frac{\Lambda}{3}}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}$$

$$R^\phi_{r\phi r} = -\frac{\partial}{\partial r}\Gamma^{\,\phi}_{r\,\phi} + \Gamma^{\,r}_{r\,r}\Gamma^{\,\phi}_{r\,\phi} - \Gamma^{\,\phi}_{r\,\phi}\Gamma^{\,\phi}_{r\,\phi} = \frac{-\frac{M}{r^3} + \frac{\Lambda}{3}}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}$$

$$\Rightarrow R_{rr} = \frac{\Lambda}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}$$

$$R^t_{\theta t\theta} = \Gamma^{\,r}_{\theta\,\theta}\Gamma^{\,t}_{t\,r} = -\frac{M}{r} + \frac{\Lambda r^2}{3}$$

$$R^r_{\theta r\theta} = \frac{\partial}{\partial r}\Gamma^{\,r}_{\theta\,\theta} + \Gamma^{\,r}_{\theta\,\theta}\Gamma^{\,r}_{r\,r} - \Gamma^{\,\theta}_{r\,\theta}\Gamma^{\,r}_{\theta\,\theta} = -\frac{M}{r} + \frac{\Lambda r^2}{3}$$

$$R^\phi_{\theta\phi\theta} = -\frac{\partial}{\partial \theta}\Gamma^{\,\phi}_{\theta\,\phi} + \Gamma^{\,r}_{\theta\,\theta}\Gamma^{\,\phi}_{r\,\phi} - \Gamma^{\,\phi}_{\theta\,\phi}\Gamma^{\,\phi}_{\theta\,\phi} = \frac{2M}{r} + \frac{\Lambda r^2}{3}$$

$$\Rightarrow R_{\theta\theta} = \Lambda r^2$$

$$\Rightarrow R_{\phi\phi} = \Lambda r^2 \sin^2\theta$$

with all terms of the form $R^\mu_{\kappa\mu\nu} = 0$ if $\kappa \neq \nu$. Therefore, $\mathbf{Ricci} = \Lambda\mathbf{g}$, and, since there are four space-time dimensions, the scalar curvature $R = g^{\mu\nu}R_{\mu\nu} = 4\Lambda$, which implies the Einstein tensor

$$\mathbf{G} = \mathbf{Ricci} - \frac{R}{2}\mathbf{g} = -\Lambda\mathbf{g}$$

demonstrating the metric is indeed a solution to the field equations in the vacuum with a cosmological constant.

Maximizing the interval along a trajectory, $\int d\tau$, yields the geodesic equations:

$$\frac{d^2t}{d\tau^2} + 2\frac{dt}{d\tau}\frac{dr}{d\tau}\Gamma^{\,t}_{t\,r} = 0$$

$$\frac{d^2r}{d\tau^2} + \left(\frac{dt}{d\tau}\right)^2 \Gamma^{\,r}_{t\,t} + \left(\frac{dr}{d\tau}\right)^2 \Gamma^{\,r}_{r\,r} + \left(\frac{d\theta}{d\tau}\right)^2 \Gamma^{\,r}_{\theta\,\theta} + \left(\frac{d\phi}{d\tau}\right)^2 \Gamma^{\,r}_{\phi\,\phi} = 0$$

$$\frac{d^2\theta}{d\tau^2} + 2\frac{dr}{d\tau}\frac{d\theta}{d\tau}\Gamma^{\,\theta}_{r\,\theta} + \left(\frac{d\phi}{d\tau}\right)^2 \Gamma^{\,\theta}_{\phi\,\phi} = 0$$

$$\frac{d^2\phi}{d\tau^2} + 2\frac{dr}{d\tau}\frac{d\phi}{d\tau}\Gamma^{\,\phi}_{r\,\phi} + 2\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}\Gamma^{\,\phi}_{\theta\,\phi} = 0$$

The θ equation is

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\,\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

which is satisfied for a planar solution, $\theta = \frac{\pi}{2}$. In this case the equations become:

$$\frac{d^2t}{d\tau^2} + 2\frac{\frac{M}{r^2} - \frac{\Lambda r}{3}}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0 \Rightarrow \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\frac{dt}{d\tau} \text{ is constant}$$

$$\frac{d^2r}{d\tau^2} + \left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right)\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{dt}{d\tau}\right)^2$$
$$- \frac{\frac{M}{r^2} - \frac{\Lambda r}{3}}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}\left(\frac{dr}{d\tau}\right)^2 - r\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} = 0 \Rightarrow r^2\frac{d\phi}{d\tau} = h \text{ is constant}$$

The r equation can be simplified using $\mathbf{g}\left(\frac{\mathbf{dx}}{\mathbf{d\tau}}, \frac{\mathbf{dx}}{\mathbf{d\tau}}\right) = -1$

$$\Rightarrow \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{\left(\frac{dr}{d\tau}\right)^2}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}} = 1 + r^2\left(\frac{d\phi}{d\tau}\right)^2$$

Substituting this in gives

$$\frac{d^2r}{d\tau^2} + \frac{M}{r^2} - \frac{\Lambda r}{3} - h^2\left(\frac{1}{r^3} - \frac{3M}{r^4}\right) = 0$$

As in the Newtonian cases, let $u = \frac{1}{r} \Rightarrow \frac{d^2u}{d\phi^2} = -\frac{1}{u^2h^2}\frac{d^2r}{d\tau^2}$, yielding

$$\frac{d^2u}{d\phi^2} + u - \frac{M}{h^2} - 3Mu^2 + \frac{\Lambda}{3h^2u^3} = 0$$

Letting $u = \frac{M}{h^2}v$ produces

$$\frac{d^2v}{d\phi^2} + v - 1 - \epsilon v^2 + \frac{\delta}{v^3} = 0$$

where $\epsilon = 3\frac{M^2}{h^2} \approx \frac{1}{1-e^2}\left(\frac{2\pi a}{T}\right)^2$ is small if the planet moves slowly compared to the speed of light, and $\delta = \frac{\Lambda h^6}{3M^4} \approx \frac{\left(1-e^2\right)^3 \Lambda T^2}{3(2\pi)^2}$ is small despite the large size of T^2 because Λ is very small. Employing the technique of multiple scales, let

$$v = \sum_{i,j=0}^{\infty} v_{ij}\epsilon^i\delta^j$$

and $\Phi_{ij} = \epsilon^i\delta^j\phi$, so by the chain rule

$$\frac{d}{d\phi} = \sum_{i,j=0}^{\infty} \frac{d\Phi_{ij}}{d\phi}\frac{\partial}{\partial\Phi_{ij}} = \sum_{i,j=0}^{\infty} \epsilon^i\delta^j\frac{\partial}{\partial\Phi_{ij}}$$

Then the equation becomes:

$$\left(\frac{\partial}{\partial\Phi_{00}} + \epsilon\frac{\partial}{\partial\Phi_{10}} + \delta\frac{\partial}{\partial\Phi_{01}} + \cdots\right)^2 (v_{00} + \epsilon v_{10} + \delta v_{01} + \cdots)$$
$$+ (v_{00} + \epsilon v_{10} + \delta v_{01} + \cdots) - 1$$
$$-\epsilon\left(v_{00} + \epsilon v_{10} + \delta v_{01} + \cdots\right)^2 + \delta\left(v_{00} + \epsilon v_{10} + \delta v_{01} + \cdots\right)^{-3} = 0$$

The order $\epsilon^0\delta^0$ equation is:

$$\frac{\partial^2 v_{00}}{\partial\Phi_{00}^2} + v_{00} - 1 = 0 \Rightarrow v_{00} = 1 + Be^{i\Phi_{00}} + \overline{B}e^{-i\Phi_{00}}$$

where B is a complex-valued function independent of Φ_{00}.

The order $\epsilon^1\delta^0$ equation is:

$$\frac{\partial^2 v_{10}}{\partial\Phi_{00}^2} + v_{10} = -2\frac{\partial^2 v_0 0}{\partial\Phi_{10}\partial\Phi_{00}} - \epsilon v_{00}^2$$

To avoid the occurrence of a secular term, the resonant terms proportional to $e^{i\Phi_{00}}$ and $e^{-i\Phi_{00}}$ need to cancel, so

$$-2i\frac{\partial B}{\partial \Phi_{10}} + 2B = 0$$

and its complex conjugate

$$2i\frac{\partial \overline{B}}{\partial \Phi_{10}} + 2\overline{B} = 0$$

Writing $B = be^{i\beta}$, and by setting the real and imaginary parts equal to zero these become

$$\frac{\partial b}{\partial \Phi_{10}} = 0$$

and

$$2b\frac{\partial \beta}{\partial \Phi_{10}} + 2b = 0$$

so b is independent of Φ_{10} and $\beta = -\Phi_{10} + \gamma$, some function independent of both Φ_{00} and Φ_{10}.

The order $\epsilon^0\delta^1$ equation is:

$$\frac{\partial^2 v_{01}}{\partial \Phi_{00}^2} + v_{01} = -2\frac{\partial^2 v_0 0}{\partial \Phi_{01}\partial \Phi_{00}} - \frac{1}{v_{00}^3}$$

Once again, to avoid the occurrence of a secular term, the resonant terms proportional to $e^{i\Phi_{00}}$ and $e^{-i\Phi_{00}}$ need to cancel. Reusing the calculation from the perturbation due to other planets, this condition becomes:

$$-2i\frac{\partial B}{\partial \Phi_{01}} + 3B\left(1 - 4B\overline{B}\right)^{-\frac{5}{2}} = 0$$

and its complex conjugate

$$2i\frac{\partial \overline{B}}{\partial \Phi_{01}} + 3\overline{B}\left(1 - 4B\overline{B}\right)^{-\frac{5}{2}} = 0$$

Writing $B = be^{-i\Phi_{10}+i\gamma}$, and by setting the real and imaginary parts equal to zero these become

$$\frac{\partial b}{\partial \Phi_{01}} = 0$$

and

$$2b\frac{\partial \gamma}{\partial \Phi_{01}} + 3b\left(1 - 4b^2\right)^{-\frac{5}{2}} = 0$$

so b is independent of Φ_{01} and $\gamma = -\frac{3}{2}\left(1-4b^2\right)^{-\frac{5}{2}}\Phi_{01}+$ some function independent of Φ_{00}, Φ_{10}, and Φ_{01}.

Concluding, to this order, it is possible to identify b with $\frac{1}{2}e$, so the expression for v is:

$$v = 1 + e\cos\left(\left(1 - \epsilon - \frac{3}{2}\delta\left(1-e^2\right)^{-\frac{5}{2}}\right)(\phi - \phi_0)\right)$$

and the angle of precession over the course of one orbit is:

$$2\pi\epsilon + 3\pi\delta\left(1-e^2\right)^{-\frac{5}{2}} = \frac{6\pi}{(1-e^2)}\left(\frac{2\pi a}{T}\right)^2 + \frac{\sqrt{1-e^2}\Lambda T^2}{4\pi}$$

The first term is the well-known relativistic correction which provided one of the key early tests for Einstein's theory and which has been abundantly verified. The second term disagrees with Rindler, but agrees with Iorio [13], neither of whom provide derivations. In any case, the currently accepted value for Λ is so small that the effect is unmeasurable on less than galactic scales. [13]

0.7 Conclusion

With the advent of large, silvered-glass reflecting telescopes and photographic plates, astronomy has become dominated by studies of distant structures and phenomena, whereas, previously, detailed observation of the solar system had been predominant. The corresponding calculations contributed to the development of perturbation theory, which unfortunately has generally rested on a plethora of seemingly random substitutions combined with *ad hoc* assumptions. Using the multiple scales technique provides a clear, consistent basis for computation, giving greater confidence in the results.

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