## Jan 16

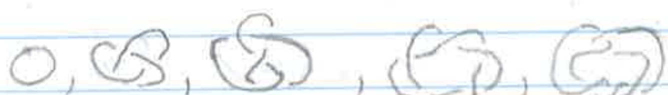
10 min | syllabus + logistics overview. describe projects + requirements
next time: PEV overview in last 10 min b/c deadlines are soon!

10 min | introductions: name, year, math interests/what you hope to get from class, non-math fact/interest

10 min | we want to study circles in 3-space think of as loop of string w/o thickness
helpful tool: projections. draw some knots on the board
define crossings: only singularities of proj'ns we'll allow
knots are equivalent if they're ambiently isotopic. we'll learn defn later – for now, demonstrate w/ physical knots.

10 min | in groups of 2-4, draw as many knots w/ 2-5 crossings as you can come up with, which are different (not ambiently isotopic)
and which have minimal # of crossings (demonstrate w/ physical knot)

10-15 min | check in w/ each group. have each group (at boards) draw a couple of the knots they found. how do you know min # of crossings?
how do you know your knots are different from ea. other?

5 min | draw some knots: 
unknot trefoil figure 8

rest of time | time to be more precise
why do we have to do this? if we allow certain irregularities we can get physically impossible knots:
- wild knots are still continuous
- shouldn't be able to pass knot thru self or pull knots to a point

def | $\mathbb{R}^n$ = n-tuples of real #s. we care most about $\mathbb{R}^1 = \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$
has a distance function $|(x_1, \ldots, x_n) - (y_1, \ldots, y_n)| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$

def | $U \subset \mathbb{R}^n$ is <u>open</u> if $\forall x \in U$, $\{y \in \mathbb{R}^n \mid |x-y| < \varepsilon\} \subset U$ for some $\varepsilon > 0$

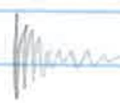
def $X \subseteq \mathbb{R}^n$, $f: X \to \mathbb{R}^m$ is <u>continuous</u> if $\forall V \subseteq \mathbb{R}^m$ open, $f^{-1}(V)$ is open in $\mathbb{R}^n$

but remember, continuous circles aren't enough (wild knots)
we want differentiability: remember SS and parameterized curves
and sfcs? we want to be able to use those ideas

def $U \subseteq \mathbb{R}^n$ open. $f: U \to \mathbb{R}^m$ <u>smooth</u> if all partial derivatives of $f$ exist
and are continuous

def $X \subseteq \mathbb{R}^n$ any set. $f: X \to \mathbb{R}^m$ <u>smooth</u> if $\exists$ $U \subseteq X$; $U$ open, extension
$F: U \to \mathbb{R}^m$ smooth, $F|_X = f$

examples of fcns from domains in $\mathbb{R}$ to ranges in $\mathbb{R}^2$:

discontinuous:  we need the open extension:  cont but not smooth: 

$\Gamma(\sin(1/x))$

def smooth $f: X \subseteq \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$ is an <u>immersion</u> if its tangent vector is never zero
smooth $f: X \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ <u>immersion</u> if the linear map $\begin{pmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \\ \partial f_3/\partial u & \partial f_3/\partial v \end{pmatrix}$ is injective (check columns lin indep)

def smooth $f: X \subseteq \mathbb{R}^{1 \text{ or } 2} \to \mathbb{R}^{2 \text{ or } 3}$ is an <u>embedding</u> if it's an immersion
which is also a bijection onto its image

nonexamples:  and  are immersions (the 6 if one end is open)
but not embeddings

 is not an immersion

now we have to talk about functions with have weirder domains
def $S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid |x - 0| = 1\}$
we care most about $S^1$, the circle
we can't get $S^1$ as the image of an embedding $\mathbb{R} \to \mathbb{R}^2$ or to $\mathbb{R}^3$
but we can think of $S^1$ in $\mathbb{R}^2$

→

hwk: will announce at end

Jan 18: start here.

def a knot is an embedding $f: S^1 \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, up to reparameterization

? explain intuitively

alternately, if you'd rather not go thru $\mathbb{R}^2$:

def a knot is an embedding $f: \mathbb{R} \to \mathbb{R}^3$ where $f(x+2\pi) = f(x)$, up to reparameterization

a knot projection is now easy: project the knot onto a plane under the map $(x_1, x_2, x_3) \mapsto (x_1, x_2)$, and remember crossing data

fact most projections are immersions $S^1 \to \mathbb{R}^2$, so a projection for us is defined as immersion

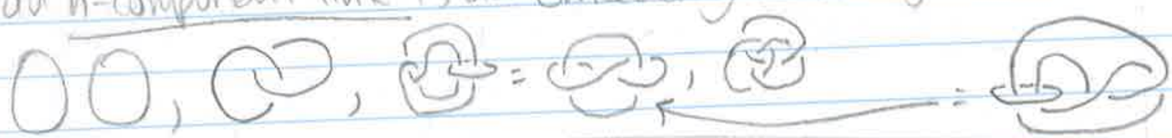
need to make our equivalence relation – corresponding the physical knot around in space – precise

def an ambient isotopy is a continuous map $F: \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ where each $F(\cdot, t)$ is continuous and a bijection with $F(\cdot, t)^{-1}$ also continuous and $F(\cdot, 0)$ is the identity on $\mathbb{R}^3$ (need this def'n to avoid pulling knots tight!)

$K_1, K_2$ are equivalent if there's an ambient isotopy $F$ with $F(K_1, 1) = K_2$

def an n-component link is an embedding of n disjoint circles in $\mathbb{R}^3$

ex



def $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

note we have $(\cos(t), \sin(t)): \mathbb{R} \to S^1$, bijection on $[0, 2\pi)$, smooth w/ smooth inverse on $[0, 2\pi)$ too

def a knot is $f = (f_1, f_2, f_3): S^1 \to \mathbb{R}^3$, which we can write as $f_i(x,y)$ where $(x,y) \in S^1$, st:

- bijection on $[0, 2\pi)$ to its image = no self crossings!
- the vector $(f_1'(t), f_2'(t), f_3'(t)) \neq (0,0,0)$ = no kinks!

non ex: if we restricted $f$ to $(-1,1)$ and used $f(t) = (t^2, t^3, 0)$

now go back to projections etc...

Jan 18: start over page. ask for @berkeley.edu emails if possible

remember to stick around for summer discussion in last 10 min!

Summer discussion:

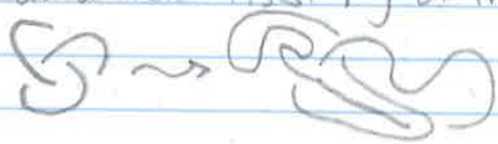
- ~~pass out info sheet +~~ tell them info is linked to website
- ask each person what they're interested in for summer + what they've done (math-related) in the past

→ they have links + know ea. other!

most of this class will be spent on understanding when knots are and aren't the same

luckily we have a relatively simple procedure to help us tell when two knots are the same, and we can tell in any projection

def a planar isotopy is an isotopy of a projection through projections
it's an ambient isotopy of the projection in $\mathbb{R}^2$

eg: 

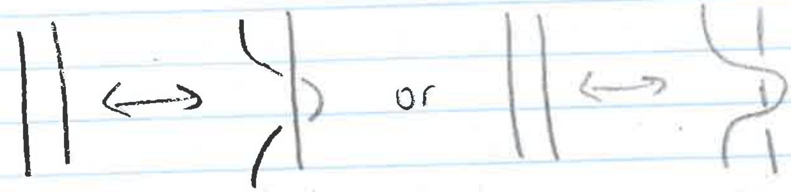
but we can clearly deform an embedding of a knot in $\mathbb{R}^3$ in worse ways (use the physical knot to show that eg proj'n of making a loop is not an isotopy thru projections - draw movie of proj'ns too:

not embedding!

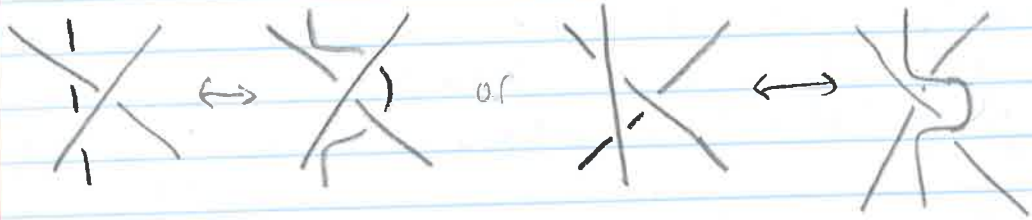
Reidemeister moves: the claim is always that if we change either side of the equivalence to the other then though we've changed the projection, the knot remains the same

RI: 

II :

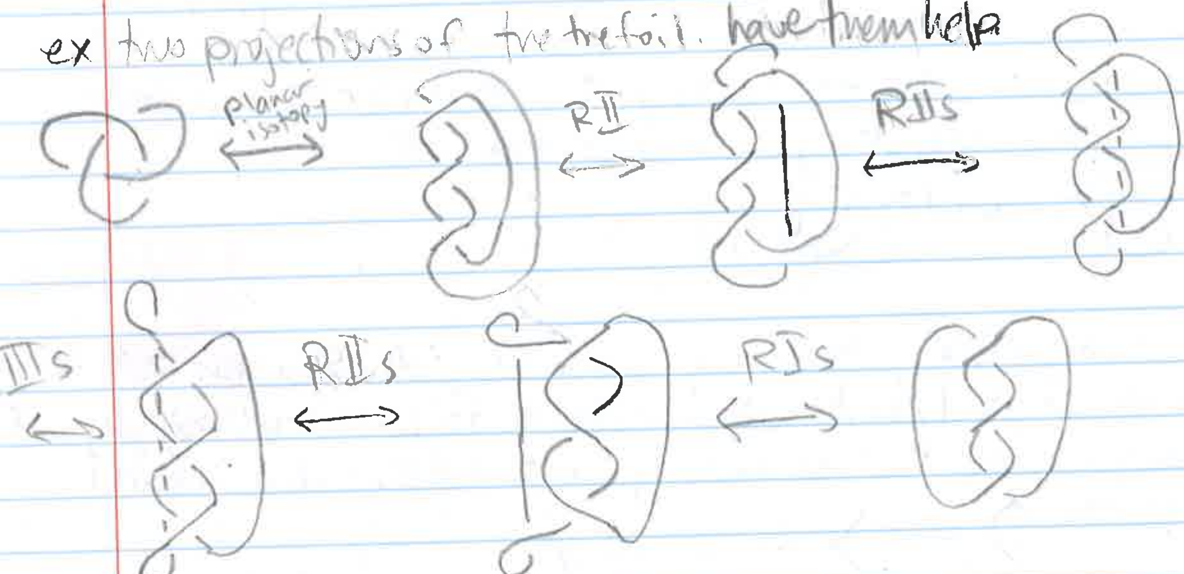



III :




thm (Reidemeister, 1926) Any two projections of the same knot are related by planar isotopies and RI–III

ex two projections of the trefoil. have them help




SHORT PROJECT IDEA: look into Reidemeister's proof
think about different dimensions eg
$S^1 \to \mathbb{R}^2$, $S^2 \to \mathbb{R}^4$

thm (Lackenby, 2015) If $K$ is the unknot, any projection of $K$ with $c$ crossings can be reduced to $O$ with at most $(236c)^{11}$ Reidemeister moves.

SHORT OR LONG PROJECT IDEA:
investigate complexity of recognizing the unknot – eg in NP, coNP

SHORT OR LONG PROJECT IDEAS:
- look into various unknotting algorithms, esp Kuperberg + Lackenby
- try to come up with your own!
- how many crossings do you have to add to unknot an unknot proj'n? we added a bunch to change the trefoil! can you bound?

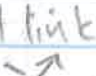
notice that just b/c there's a way, doesn't mean it's obvious or short! moral: knots are not easy


needs more crossings to be added to unknot!
⟹ computational complexity

links

def an n-component link is a map f of a disjoint union of n circles (see prev pg for examples)

def a link is splittable if its components can lie on either side of a plane in $\mathbb{R}^3$
it's often hard to tell!

def linking #: let M, N be the components of a 2-component link L choose an orientation on each. count crossings:  $=+1$,  $=-1$ sum and divide by 2. do not count crossings b/twn M and itself or N and itself. the abs. val. of the # is $lk(M,N)$ (take $|\cdot|$ b/c orientations ⟺ sign)

ex O O has $lk=0$,  has $lk=1$,  has $lk=2$

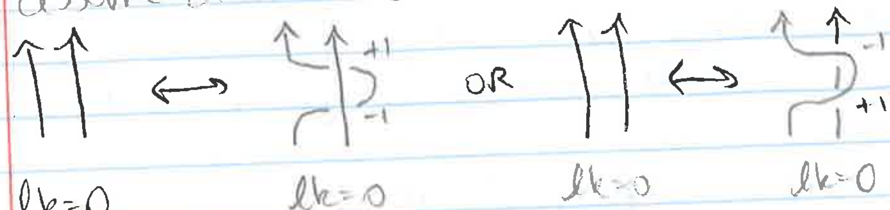
the thm on Reidemeister moves works for linking # as well. that's how we'll show that lk is an inv't of a 2-component link

claim: even though we defined lk using a projection, it's a property of the link. such a property is called an invariant. the idea is it's invariant of choices (the way the circles sit in $\mathbb{R}^3$ and the projection chosen to $\mathbb{R}^2$)

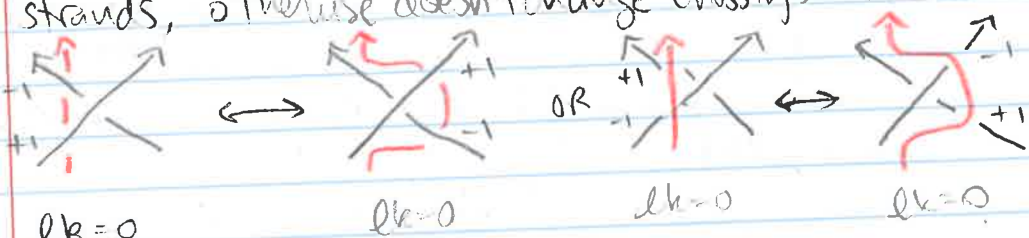
pf show lk unchanged under each RI-III
(planar isotopies don't change lk b/c don't change crossings) ⟶

RI | can't change crossings btwn two components

RII | assume different strands

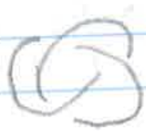



RIII | assume the strand which moves is different from the crossed strands, otherwise doesn't change crossings btwn diff. components




ex | compute linking #s of

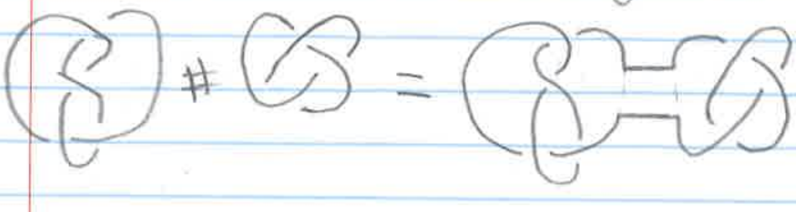

whoops! linking # can't tell Whitehead link from unlink
also: Borromean rings - any 2 components aren't linked
⇒ we need more invariants!

hw: show  and  same knot by finding a series of Reidemeister moves. be careful!

start w/ pf of lk inv't
say website updated – 2 absences ok. HW C DUE my bad!
discuss project guidelines. say that suggestions are ongoing
also look to add cons slides b/c I'm trying not to duplicate

## Jan 23
**composition**

def — let J and K be knots. denote their __composition__ by "J#K":
choose an arc in each on "outside" of each proj'n and avoiding crossings
choose proj'ns to not overlap
remove arcs and reconnect by two new arcs which don't cross anything



≠  eg

def — a knot is __composite__ if it can be expressed as J#K for J, K ≠ U
J, K are __factors__

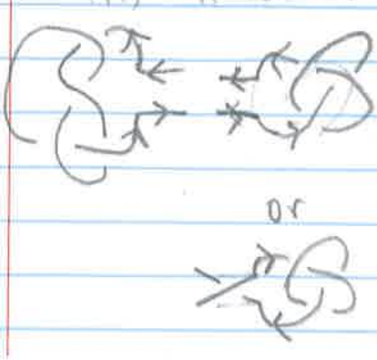
note — K#U = K

def — a knot is __prime__ if it is not composite
fact — both trefoil and fig 8 are prime – not obvious!

does the unknot have to be prime? what about trefoils, fig 8?
yes! (we'll prove this in about a month)

notice that J#K depends on choices: of projections, of arcs
claim — # is well-defined for oriented knots, if we force the orientations to match
def — __orientation__ on a knot is a direction

pf — what is "# for oriented knots"? choose proj'n so orientation extends over the resulting knot



choice of arcs ⟷ choice of pt on each knot by making arcs into a ribbon, which is indep of proj'n b/c can isotope

$6_{8}^{8_{17}}$

however, you will have to draw big knots to see an example where the different orientations give different compositions

def a knot is invertible/reversible if it's isotopic to itself w/ orientation switched

the first non-invertible knot has 8 crossings ($8_{17}$)

Tabulating knots

def the crossing number of K is the least number of crossings in any projection of K

def the mirror image of K is the reflection of K thru a plane in $\mathbb{R}^3$; alt, change all crossings in a diagram of K

def K is amphicheiral (chemists: achiral) if it's isotopic to its mirror image

fact fig-8 is amphicheiral

the knot table is organized by crossing number
only prime knots are listed
and mirror images are not listed

Tricolorability: our first invariant of knots!

def a strand of a proj'n is a piece from one undercrossing to another w/ only overcrossings btwn

def a tricoloring of a proj'n is a coloring of the strands by R, G, B where at each crossing, 3 or 0 colors come together

def a knot/link is tricolorable if ∃ proj'n which can be tricolored with at least 2 of the colors