Relations.

(1) (a) transitive, antisymmetric
(b) reflexive, transitive, antisymmetric
(c) symmetric, reflexive, transitive
(d) antisymmetric
(e) antisymmetric
(f) symmetric

(2) (a) $\frac{2^{n+\frac{n(n-1)}{2}}}{2^{n^2}}$
(b) $\frac{2^{(n-1)^2}}{2^{n^2}}$
(c) $\frac{2^{\frac{n(n-1)}{2}}}{2^{n^2}}$

(3) (a) $x \neq y$
(b) This one maybe isn't the best, but it works: let R be the relation on $\mathbb{R}$ where $(x, y) \in R$ if $x \neq y$ or $x = y^2$.

(4) (a) T: (a, b) is in both iff (b, a) is in both
(b) F: $x \neq y$ is the union of $x < y$ and $x > y$
(c) T: if (a, b) and (b, c) are in both then by transitivity of each, (a, c) is in both
(d) F: $R_1 = \{(a, b)\}$ and $R_2 = \{(b, c)\}$ for no two of a, b, c equal

(5) (a) reflexivity requires $(a, a) \in R$ for *all* $a \in A$
(b) $(x, y) \in R$ when $x \leq y$ and $y \leq 2$

Representing Relations.

(1) (a) $\{(1, 1), (1, 3), (3, 1), (2, 2), (3, 3)\}$
(b) $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 1), (3, 2)\}$
(c) $\{(1, 2), (2, 3), (3, 1)\}$

(2) (a) transitive, symmetric, reflexive
(b) transitive
(c) antisymmetric

(3) $M_{R^{-1}} = M_R^T$, and you can obtain $M_{\bar{R}}$ from M_R by switching all the 1s to 0s and all the 0s to 1s.

Closures.

(1) For the symmetric closures, draw all the reverse arrows. For the transitive closures, complete all the triangles.

(2) (a) $\neq$
(b) $\gcd(a, b) = \min\{a, b\}$ or $\mathrm{lcm}(a, b) = \max\{a, b\}$
(c) three legs of the "X" formed by $|x| = |y|$, excluding the leg where both are negative

(3) (a) "is an ancestor of"
(b) $x - y \in \mathbb{Z}$
(c) "one can get from x to y via flying"

(4) Note $t(R)$ is what we're calling R^*
(a) T: $\Delta \subseteq R \subseteq R^*$

(b) F: any closed path will produce a loop, e.g. if we have $(1,2)$ and $(2,1)$ we'll get the loops $(1,1)$ and $(2,2)$
(c) T: $(x,y) \in R^*$ are connected by a path; flip all elements in the path and it's still a path in R by symmetry of R

Equivalence Relations.

(1) (a) not symmetric, therefore not an equivalence relation
(b) equivalence classes are congruence classes mod n
(c) equivalence classes are families
(d) equivalence classes are social circles
(e) equivalence classes are roommates
(f) not symmetric, therefore not an equivalence relation: $(a,b)R(0,0)$ but we cannot switch the order of (a,b) and $(0,0)$
(2) $[a+c]_R = [b+d]_R$ if and only if $a+c \equiv b+d \mod 3$, which is true
(3) $2^{n+\frac{n(n-1)}{2}}$
(4) (a) F: on the set $\{a,b,c\}$, $R_1 = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$ and $R_2 = \{(a,a),(b,b),(c,c),(b,c),(c,b)\}$ are equivalence relations, but their union is not (check!)
(b) T: check that the intersection of symmetric relations is symmetric, the intersection of reflexive relations is reflexive, and the intersection of transitive relations is transitive; alternately think about partitions

Graphs.

(1) (a) pseudograph
(b) directed graph
(c) multigraph
(2) (a) degrees: 0, 1, 2, 3, 4; one isolated node (lower left), one pendant node (lower middle); the sum of the degrees is 10 and the number of edges is 5
(b) in-/out-degrees, in clockwise order: 0/1, 1/2, 2/0, 2/1, 1/2; no isolated nodes and one pendant node (upper left); the sum of the in and out degrees is 12 and there are 6 edges
(c) you can do this one ;)
(3) (a) 4
(b) $\binom{6}{2} = 15$
(c) n
(d) $\binom{n}{2}$
(e) don't worry about this one
(4) no, because the sum of the degrees would have to be odd
(5) One such graph can be described as follows: draw a pentagon with a five-pointed star in it. Then connect one vertex of the pentagon to two vertices in the star, and repeat with the other four vertices in the pentagon in the same pattern.
(6) Solve this by attempting to color them with two colors. If you find a cycle of odd length this will be impossible.
(a) Start at the upper left and color the outer square red/blue/red/blue in clockwise order. Then start at the upper left of the inner square and color the inner square blue/red/blue/red in clockwise order.
(b) This one isn't bipartite – try to find an odd length cycle. Why is this a problem?

(c) Color the isolated vertex whatever color you want, and color the vertices in the K_2 opposite colors. Start at the pendant vertex and color it red, then color the vertex it's connected to blue. Color the three next vertices red, and the remaining vertex blue.

(7) (a) $n = 2$
(b) n even
(c) one partition is by the parity of the number of 1s

(8) (a) F: simply add an edge between any two vertices in the same part of the partition
(b) T: there are only going to be fewer edges to mess up the partition

(9) With zero vertices: the empty graph. With one vertex: each isolated vertex. With two vertices: each pair of vertices without an edge, and each pair with an edge. With three vertices: all vertices isolated, any of the 3 choices of edges, and then K_3 itself.

(10) Let v be a non-isolated vertex of G with $|V| = k + 1$. Consider $N(v)$, the set of vertices adjacent to v. If there is an edge between any two vertices in $N(v)$ then we are done, as the path from v to the first vertex, to the second vertex, and then back to v forms a cycle of the type we're looking for.

Now assume there are no edges between any vertices in $N(v)$. Since each vertex in $N(v)$ has at least two neighbors, G has more vertices. Let $N(N(v))$ be the set of vertices adjacent to vertices in $N(v)$. Again, if there is an edge between any two of the vertices in $N(N(v))$ then we're done; we're also done if any two or more elements of $N(N(v))$ neighbor the same element in $N(v)$.

Otherwise, we can keep going, because everything in $N(N(v))$ has only one edge accounted for. If we can keep repeating this procedure we'll end up with an infinite amount of vertices, which we don't have. So the process has to stop at some point.

Other idea: Induct on the number of vertices. Base case is $|V| = 3$. Inductive step: let $v \in V$. If there are any edges between neighbors of v then we're done, so assume there are not. Consider the graph G' which is G with v removed and $N(v)$, the neighbors of v, replaced with the complete graph $K_{|N(v)|}$. This is a graph whose edges all have an even number of neighbors; by the inductive hypothesis there is a path from any vertex back to itself using no single edge twice. We can turn these paths in G' to paths in G by using the same path off the complete graph part, and any edge in the complete graph part can be replaced by the pair of edges which go through v.

This takes care of all paths from vertices which aren't v to themselves; in order to take care of paths from v to v, simply pick a different vertex v' to replace with $K_{|N(v')|}$.