DIFFEOMORPHISMS OF SURFACES AND SMOOTH 4-MANIFOLDS

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Motivation: Lefschetz Fibrations on Smooth 4-Manifolds

There are a lot of good reasons to think about mapping class groups and lots of people, including but not limited to hyperbolic geometers, geometric group theorists, Teichmüller theorists, symplectic geometers, and 4-manifold topologists study them. Here are some reasons why I'm interested in mapping class groups.

Throughout, X will be a closed, smooth, oriented 4-manifold. (One can think about what the analogous statements should be in the case when X is only compact or is open but "standard at infinity.") Much of this section is inspired by [1].

Definition 1. *A* ***Lefschetz fibration*** *is a smooth map $f : X \to S^2$ whose critical points form a finite set and can be represented in local orientation-preserving complex coordinates by $(z_1, z_2) \mapsto z_1^2 + z_2^2$.*

We can, after perhaps perturbing f slightly, assume without loss of generality that f is injective on its critical points.

DRAW: I'll draw a schematic picture of what your idea of a neighborhood of such a singularity could be. Note that since X is closed, Σ is also closed.

Now fix $q \in S^2 \backslash Crit(f)$ and let $\Sigma = f^{-1}(q)$ be the fiber above q. Consider the **monodromy homomorphism**

$$m : \pi_1(S^2 \backslash Crit(f), q) \to \mathrm{Mod}(\Sigma)$$

What does this mean? Firstly, $\mathrm{Mod}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$, where Homeo^+ refers to orientation-preserving homeomorphisms and Homeo_0 refers to those homotopic to the identity. However, $\mathrm{Mod}(\Sigma)$ is isomorphic to $\mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0(\Sigma)$ because on compact surfaces, homeomorphisms are isotopic to diffeomorphisms, and on closed surfaces, homotopic homeomorphisms are isotopic (see [4] §1.4).

Secondly, how do we define m? $\pi_1(S^2 \backslash Crit(f), q)$ consists of classes of paths γ from an interval. Put a horizontal distribution on the fibration over the image of γ (for example, by taking the orthogonal complement of the fiber directions using some Riemannian metric) and define m to be the isotopy class of the diffeomorphism of Σ induced by parallel transport in the pullback bundle $\gamma^* f$. The result is independent of the horizontal distribution as horizontal distributions correspond to 2-forms on the base of the pullback bundle, and here there are none.

How can we get our hands on this monodromy? Notice that in real coordinates the smooth fibers $\{z_1^2 + z_2^2 = \epsilon > 0\}$ are given by the set of x_i and y_i for which

$$x_1^2 - y_1^2 + x_2^2 - y_2^2 = \epsilon \text{ and } 2x_1y_1 + 2x_2y_2 = 0$$

and that the set $\{x_1^2 + x_2^2 = \epsilon, y_i = 0\}$ shrinks to the critical point $z_1 = z_2 = 0$ of f as ϵ goes to zero. (That's pretty vague: what we're really doing is picking the path between ϵ and zero which stays on the real axis in $\mathbb{C}$

and considering the parallel transport of the circle above this path with respect to the horizontal distribution given by the totally real subspaces $\mathbb{R}^2$ of $\mathbb{C}^2$.) By symmetry of the local model there's an analogous circle in each nearby fiber, called the **vanishing cycle** of that Lefschetz critical point. One can show that the monodromy of a loop in the base homotopic to a single critical value is a right-handed Dehn twist in the corresponding vanishing cycle.

DRAW: I'll take a moment to draw a right-handed Dehn twist. Note that it's compactly supported in a normal neighborhood of the vanishing cycle.

What do Lefschetz fibrations have to do with the word problem in mapping class groups? Choose a collection $\eta_1, \ldots, \eta_r$ of disjoint arcs joining q to the critical values p_i of f and draw loops γ_i based at q along η_i around each critical value.

DRAW: This is what I mean.

DRAW: Here's a picture to convince you that $\Pi\gamma_i = id$ in $\pi_1(S^2 \backslash Crit(f), q)$.

Denote the Dehn twist about the vanishing cycle associated to p_i by τ_i. Since m is a homomorphism (as usual in monodromy homomorphisms, concatenation of paths just means you keep transporting along the horizontal distribution, which is precisely composition of the associated fiber diffeomorphisms), $\tau_1 \cdots \tau_r = id$ in $\text{Mod}(\Sigma)$!

So at this point we know that Lefschetz fibrations with fiber Σ determine factorizations of the identity in $\text{Mod}(\Sigma)$. The converse is also true: inspecting the local model for a Lefschetz singularity shows that projection to the real value is Morse, and Lefschetz critical points are index two, so a Lefschetz singularity corresponds to adding a 2-handle along the vanishing cycle to the boundary of a preimage of a small disk about q. Note that the order of the factorization matters, and has to correspond with the order of the paths coming in to q from the p_is. This gives us the fibration above a disk containing q and the p_is; what remains is to glue in $\Sigma \times \mathbb{D}^2$ along the common boundary $\Sigma \times S^1$. There is some ambiguity in doing this as for $\Sigma = S^2, T^2$, $\text{Diff}(\Sigma)$ has π_1, but when the genus of Σ is ≥ 2 there is no further ambiguity and the factorization determines the fibration $f : X \to S^2$.

Furthermore, there are some equivalences on either side which also respect the correspondence. Nobody forced us to pick the η_is we did, and when we change them by **Hurwitz moves**...

DRAW: A Hurwitz move. This changes the data of the fibration but doesn't change the total space. In fact, it only changes the fibration up to isotopy. You've basically just slanted the direction which is "up" in a small tube where you're moving the chart for the local picture of the singularity.

...(this is the action of the braid group on the γ_is, but don't worry about that, just look at the picture) we get **Hurwitz equivalence** on sets of Dehn twists on the mapping class group identity factorizations side:

$$(\tau_1, \ldots, \tau_i, \tau_{i+1}, \ldots, \tau_r) \sim (\tau_1, \ldots, \tau_i\tau_{i+1}, \tau_i^{-1}, \tau_i, \ldots, \tau_r)$$

(You can see that their product is of course still the identity, but the product is not the point, it's the factorization.)

Finally, the dependence on q should be removed, and since changing the basepoint of a fundamental group corresponds to conjugation in the group, we also allow factorizations to be considered equivalent when they're **globally conjugate**:

$$(\tau_1, \ldots, \tau_r) \sim (\gamma\tau_1\gamma^{-1}, \ldots, \gamma\tau_r\gamma^{-1})$$

The ultimate equivalence is then

{Lefschetz fibrations}/isotopy $\overset{1-1}{\leftrightarrow}$ {factorizations of id in Mod(Σ)}/Hurwitz equivalence and global conjugation

Okay, that's great. But why are Lefschetz fibrations over S^2 interesting? What does "interesting" even mean? One possible answer is that Lefschetz fibrations are prevalent and tightly intertwined with an interesting geometric structure on 4-manifolds: symplectic structures.

Theorem 1. *(Gompf) Let $f : X \to S^2$ be a Lefschetz fibration with the class of the fiber nontrivial in $H_2(X;\mathbb{R})$. Then X admits a symplectic structure, unique up to deformation, with symplectic fibers.*

Theorem 2. *(Donaldson) A compact symplectic 4-manifold (X, ω) carries a Lefschetz fibration on some symplectic blowup $X\#k\overline{\mathbb{CP}}^2$.*

The Word Problem in Mapping Class Groups

If a is not isotopic to a point, then the Dehn twist about a is a nontrivial element of Mod(Σ). If a is nonseparating, know that there's always a diffeomorphism of Σ so that a is standard. Then consider the action of τ_a on the standard b with $i(a, b) = 1$; b and $\tau_a(b)$ are not isotopic as they're not even cohomologous. If a is separating, you can consider the action of τ_a on the standard b with $i(a, b) = 2$; $\tau_a(b)$ is not isotopic to b by seeing that it satisfies the bigon criterion with respect to a so is in minimal position (unfortunately I will not show that no bigons forces minimal position – minimal position is when the cardinality of the set of intersections of a and b achieves the minimum over all possible isotopic copies of a and b), and then checking the intersection number (intersection number is the minimum over all homotopy/isotopy class representatives for each curve of the intersection numbers of those curves, so is an isotopy invariant).

Moreover, Dehn twists have infinite order. You can see this by checking the intersection number of $\tau_a^k(b)$ with b for a, b as above; it's going to be $|k|i(a, b)^2$.

Now the statement of this theorem at least sounds plausible:

Theorem 3. *For $g \geq 0$ the group Mod(Σ_g) is generated by finitely many Dehn twists about nonseparating simple closed curves.*

DRAW: Here's an explicit set of generators.

I'm not going to talk about presentations, but with finite generation alone we can actually understand how to solve the word problem for mapping class groups! This is exciting because it allows us to think that asking whether a product of Dehn twists obtained from the monodromies of a Lefschetz fibration is the identity or not is a reasonable question!

The general idea to solve the word problem is as follows:

- Take a word w in Dehn twists and a representative ϕ of the product of Dehn twists making up w.
- Find a set $\gamma_1, \ldots, \gamma_n$ of embedded closed curves in Σ which satisfy:
  - the γ_i are in pairwise minimal position
  - they're pairwise nonisotopic

- for distinct i, j, k at least one of the pairwise intersections of the corresponding curves is empty
- $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ for some permutation $\sigma \in S_n$
- the γ_is **fill** Σ, that is, $\Sigma \backslash (\sqcup \gamma_i)$ is a disjoint union of disks

- Choosing the set of curves γ_i properly is always possible. The choice requires the word w and it's finiteness, not just ϕ!

- **Theorem 4.** *(Alexander Method) If ϕ_* fixes each vertex and each edge of the graph Γ (vertices at the intersection points of the γ_is, edges given by $\sqcup \gamma_i \backslash$intersections) along with orientations then ϕ is isotopic to the identity. If not, ϕ has a nontrivial power ϕ^r which is isotopic to the identity.*

(Further, when ϕ is not isotopic to the identity, one can show the stronger statement that it's isotopic to a finite order homeomorphism.)

The idea of the proof is as follows: first, we show that the individual isotopies $\phi(\gamma_i) \sim \gamma_{\sigma(i)}$ extend to a global isotopy of the unions of curves. Then we can think of ϕ as an automorphism of Γ. Next we apply the fact that the automorphism group of a finite graph is finite to get the power r, if necessary, for which ϕ^r is the identity on Γ. Finally we apply the Alexander Lemma to $\Sigma \backslash \sqcup \gamma_i$ to show that ϕ^r is isotopic to the identity on the complement of the γ_is.

Before the next lemma, know that we define the mapping class group of a surface-with-boundary to consist of those homeomorphisms which are the identity near the boundary modulo those homotopies which fix the boundary. We can apply a result on $\text{Mod}(\mathbb{D}^2)$ in the above because we know that fixing Γ means fixing the boundary of all the disks determined by the filling set $\{\gamma_i\}$.

Lemma 1. *(Alexander Lemma) Mod$(\mathbb{D}^2)$ is trivial.*

Proof. Identify $\mathbb{D}^2$ with the closed unit disk in $\mathbb{R}^2$. Let $\phi : \mathbb{D}^2 \to \mathbb{D}^2$ be a homeomorphism equal to the identity on $\partial \mathbb{D}^2$. The isotopy for $0 \leq t < 1$ is given by

$$F(x,t) = \begin{cases} (1-t)\phi\left(\frac{x}{1-t}\right) & 0 \leq |x| < 1-t \\ x & 1-t \leq |x| \leq 1 \end{cases}$$

and $F(x, 1) = id_{\mathbb{D}^2}$. □

The idea of the proof is to squeeze the homeomorphism down to a point. This proof shows the strength of the fact that $\text{Homeo}^+(\Sigma)/\text{Homeo}_0(\Sigma) \cong \text{Diff}^+(\Sigma)/\text{Diff}_0(\Sigma)$ – we can use isotopies (homotopy through homeomorphisms) of homeomorphisms freely without worrying about smoothness.

Problems

Difficulty of the word problem. Knowing the word problem is solvable is great. Knowing it's solvable in a certain amount of time is even better. Mosher [5] showed that $\text{Mod}(\Sigma_g)$ is automatic, which in particular means that it has a word problem solvable in quadratic time. There are reasons to hope – because of hyperbolic geometry which I don't understand – that there could be a sub-quadratic time algorithm to solve the word problem. A proposed time is $n \log n$ [3].

Writing a diffeomorphism in standard form. Even if you have a solvable word problem you're out of luck if you have a diffeomorphism which you can't write in terms of the generators you like. An example that I'm interested in shows up in [2], where you consider a diffeomorphism of a genus $g-1$ surface induced by a product of Dehn twists on a genus g surface. The surfaces appear as fibers of a Morse function with an index two critical point whose corresponding handle is attached along c, and the diffeomorphism is obtained by pushing the product of Dehn twists across the handle attachment via parallel transport along some horizontal distribution on the total space of the Morse function (that is, transverse to the fibers). It is only possible to do this if the product of all the Dehn twists fixes c, but it is possible for a diffeomorphism which fixes c to factor into Dehn twists with some components of the factorization Dehn twists about curves intersecting c. Therefore simply doing the naive thing – factoring the map on Σ_{g-1} as the product of the Dehn twists about the parallel-transported curves – is not always possible, and we have to find another factorization.

There are probably many more examples of diffeomorphisms of surfaces which are obtained from some geometric method, perhaps also from four-manifold topology as the one I just described is.

Isotopy and surgery as moves in the mapping class group. These problems are from [1].

Based on the first section, you can see that telling whether two Lefschetz fibrations are isotopic is equivalent to telling when two factorizations in $\mathrm{Mod}(\Sigma)$ are equivalent up to Hurwitz moves and global conjugation. So the question is: is this second problem decidable? (Is it or is it not possible to construct an algorithm for this problem?) Whether or not it is, are there any necessary or sufficient criteria for equivalence or inequivalence up to Hurwitz moves and global conjugation?

The second problem involves **Luttinger surgery**: cutting out a tubular neighborhood of a Lagrangian torus in X which is foliated by parallel Lagrangian tori, and gluing it back in via a symplectomorphism which wraps the meridian around the torus in the direction of some given closed loop on the torus, while not affecting the longitudes. For Lefschetz fibrations we consider tori which are fibered above embedded loops γ in $S^2 \backslash Crit(f)$, whose fibers are embedded closed loops in the fibers of f and invariant under the monodromy along γ. You can build such a torus by parallel transporting a loop α in the fiber along γ. The mapping class group manipulation corresponding to Luttinger surgery is conjugating all the Dehn twists associated to critical points in the interior of γ by τ_α. You have to choose α so that τ_α commutes with the product of the Dehn twists involved so that you get another factorization of the identity at the end.

So the question is: given two factorizations of the identity in $\mathrm{Mod}(\Sigma)$ as a product of Dehn twists along nonseparating curves, is it always (when Euler characteristic and signature of the total spaces are the same) possible to get one from the other by Hurwitz moves and partial conjugations? This is the same question as: given two compact integral symplectic 4-manifolds with the same $(c_1^2, c_2, [\omega]^2, c_1 \cdot [\omega])$, are they related by a sequence of Luttinger surgeries?

References

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