14.5 Problem 53 *If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then show*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

Since terms on the right hand side are calculated in terms of the left hand side terms, the strategy is to calculate the right hand side and show that it equals the left hand side.

To do this, we need $\frac{\partial z}{\partial r}$, $\frac{\partial^2 z}{\partial r^2}$, and $\frac{\partial^2 z}{\partial \theta^2}$. My calculations in section for the first two were correct. However, in my 8-9 section, I made a mistake calculating the third derivative. Here is a corrected calculation.

Using a straightforward application of the chain rule,

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

Now to take the second derivative,

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\frac{\partial z}{\partial \theta} \\ &= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\right) \\ &= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\right)\end{aligned}$$

We have to be careful here since $\frac{\partial x}{\partial \theta}$ and $\frac{\partial y}{\partial \theta}$ are functions of θ, so we have to use the product rule – this was not the case when calculating $\frac{\partial^2 z}{\partial r^2}$ since the terms $\frac{\partial x}{\partial r}$ and $\frac{\partial y}{\partial r}$ are not functions of r, so they could be treated as constants. We get

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\right) \\ &= \left(\frac{\partial}{\partial \theta}\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial x}\left(\frac{\partial}{\partial \theta}\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial}{\partial \theta}\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial y}\left(\frac{\partial}{\partial \theta}\frac{\partial y}{\partial \theta}\right) \\ &= \left(\frac{\partial}{\partial x}\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial z}{\partial x}\frac{\partial y}{\partial \theta}\right)\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial \theta^2} + \left(\frac{\partial}{\partial x}\frac{\partial z}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\right)\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial \theta^2} \\ &= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial \theta^2} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial \theta}\right)^2 + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial \theta^2} \\ &= \frac{\partial^2 z}{\partial x^2}(-r\sin\theta)^2 + \frac{\partial^2 z}{\partial x \partial y}(r\cos\theta)(-r\sin\theta) + \frac{\partial z}{\partial x}(-r\cos\theta) + \frac{\partial^2 z}{\partial y \partial x}(-r\sin\theta)(r\cos\theta) \\ &\quad + \frac{\partial^2 z}{\partial y^2}(r\cos\theta)^2 + \frac{\partial z}{\partial y}(-\sin\theta)\end{aligned}$$

You can plug this in as $\frac{\partial^2 z}{\partial \theta^2}$ into the expression $\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$ (along with the expressions for $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$ obtained in section, which are correct) to get the result.