Math 53 Discussion: Review for Final

Vector fields, line integrals. §16.1–16.2

- A vector field is a function on the plane or in three-dimensional space. It's a function which assigns a vector at each point in its domain. For example, $\overrightarrow{F}(x, y, z)$ could describe the direction the wind is traveling at (x, y, z).
- Line integrals allow us to integrate a scalar function or vector function over a curve C, by parametrizing the curve and reducing to an integral in one variable.
- Integrating a scalar function (e.g. if C describes a thin wire with linear density $f(x, y, z)$ then we integrate f over C to get the mass of the wire):

$$\int_C f\, ds$$

  where s is arc length.
- Integrating a vector field (e.g. to find the work done by the force field $\overrightarrow{F}$ over the curve C):

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$$

- Method:
  - Parametrize the curve C as $\overrightarrow{r}(t) = \langle x(t), y(t), z(t)\rangle$, or only in x and y if it's a planar curve.
  - Determine what type of integral you're doing: if you're integrating a scalar then take the **magnitude** $|r'(t)|$ to get

$$ds = |r'(t)|\, dt$$

    If you're integrating a vector field, then take the **dot product** of $\overrightarrow{F}$ with

$$d\overrightarrow{r} = \overrightarrow{r}'(t)dt$$

  - Plug everything into the integral $\int_C f\ \ ds$ or $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ to get an integral in t, which can then be evaluated.

The fundamental theorem of line integrals. §16.3

- For conservative vector fields $\overrightarrow{F} = \nabla f$ only we have that line integrals are independent of path:

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C \nabla f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a))$$

- In particular, if we have a closed curve, meaning $\overrightarrow{r}(a) = \overrightarrow{r}(b)$, the line integral is zero.
- If $\overrightarrow{F} = \langle P, Q\rangle$ and $\partial Q/\partial x = \partial P/\partial y$ on a simply connected region, then $\overrightarrow{F}$ is conservative on that region. This is a special case of checking that $\text{curl}(\overrightarrow{F}) = \overrightarrow{0}$ on a simply connected region.

Green's theorem. §16.4

- C is a simple closed, positively oriented curve, enclosing region D, and $\overrightarrow{F} = \langle P, Q\rangle$ has continuous partial derivatives on an open region containing D then:

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA = \int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C Pdx + Qdy$$

Curl and divergence. §16.5

- Let $\overrightarrow{F} = \langle P, Q, R \rangle$. Then
$$\text{curl}(\overrightarrow{F}) = \nabla \times \overrightarrow{F} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{pmatrix}$$

- Thinking again of $\overrightarrow{F}$ as fluid flow, curl($\overrightarrow{F}$) points in the direction of the axis of rotation of the fluid. A larger magnitude for curl means faster rotation. curl($\overrightarrow{F}$) $= \overrightarrow{0}$ means $\overrightarrow{F}$ is irrotational.

- $$\text{div}(\overrightarrow{F}) = \nabla \cdot \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- Thinking of $\overrightarrow{F}$ as describing fluid flow, divergence of $\overrightarrow{F}$ at a point measures the amount of fluid flowing in or out at that point. Positive divergence means the net flow is outwards (source) and negative divergence means the net flow is inwards (sink). div($\overrightarrow{F}$) $= 0$ means $\overrightarrow{F}$ is incompressible.

Parametrized surfaces, Surface integrals. §16.6–16.7

- We have seen two types of surface integrals. We can integrate a scalar function or we can integrate a vector function.

- Method:

  1. Parametrize S as $\overrightarrow{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, find the bounds for u, v. Here are some examples.

     Graph of a function $z = f(x, y)$: x and y are the parameters.
     $$\overrightarrow{r}(x, y) = \langle x, y, f(x, y) \rangle$$
     The domain will be the "shadow" of the surface in the xy-plane.

     A sphere of radius a: ϕ and θ are the parameters.
     $$\overrightarrow{r}(\phi, \theta) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle$$
     You determine the bounds on ϕ and θ depending on what portion of the surface of the sphere you want.

     A cylinder of radius a along z-axis: the height z and the polar angle θ are the parameters.
     $$\overrightarrow{r}(\theta, z) = \langle a \cos\theta, a \sin\theta, z \rangle$$

  2. Compute $\overrightarrow{r}_u \times \overrightarrow{r}_v$. Then $\overrightarrow{n}\, dS = \pm \overrightarrow{r}_u \times \overrightarrow{r}_v \, du \, dv$. Whatever your parameters u and v are, you only multiply the cross product by $du\, dv$. If your parameters are r and θ, you will only multiply by $dr\, d\theta$. However, if your parameters are x and y and you get a double integral involving $dx\, dy$ and then want to convert into polars, you will need to use $r\, dr\, d\theta$.

     Graph of a function:
     $$\overrightarrow{r}_x \times \overrightarrow{r}_y = \langle -f_x, -f_y, 1 \rangle$$

     A sphere of radius a:
     $$\overrightarrow{r}_\phi \times \overrightarrow{r}_\theta = \left\langle a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \sin\phi \cos\phi \right\rangle$$

Or a geometric argument: the normal is $\frac{1}{a}\langle x, y, z\rangle$ and the area element on the sphere is $dS = a^2 \sin\phi \, d\phi \, d\theta$ so multiplying these together gives

$$d\overrightarrow{S} = a \sin\phi \langle x, y, z\rangle \; d\phi \, d\theta$$

where x, y and z are in terms of ϕ and θ.

A cylinder of radius a along z-axis:

$$\overrightarrow{r}_\theta \times \overrightarrow{r}_z = \langle a\cos\theta, a\sin\theta, 0\rangle$$

Or a geometric argument: the normal is $\frac{1}{a}\langle x, y, 0\rangle$ and $dS = a \, dz \, d\theta$ so multiplying together gives

$$d\overrightarrow{S} = \langle x, y, 0\rangle \, dz \, d\theta = \langle a\cos\theta, a\sin\theta, 0\rangle \, dz \, d\theta$$

3. For integrals of a **scalar function** f: The integrand will be $f \, dS$ so we take the **magnitude**

$$dS = |\overrightarrow{r}_u \times \overrightarrow{r}_v| \, du \, dv$$

For integrals of a **vector function** $\overrightarrow{F}$: The integrand will be $\overrightarrow{F} \cdot d\overrightarrow{S}$ so we take the **dot product with the vector**

$$d\overrightarrow{S} = \pm \overrightarrow{r}_u \times \overrightarrow{r}_v \, du \, dv$$

- Surface area from §15.6 is a special case of this. In that section, we were dealing only with surfaces that were graphs of functions (not all surfaces are graphs of functions, e.g. a sphere). We were also dealing only with integrals of scalar functions, namely integrating 1. So the dS in that case is $\sqrt{f_x^2 + f_y^2 + 1}$ and the surface area is

$$\int\int_S 1 \, dS = \int\int_D \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$$

where S is the surface and D is the domain in x and y.

- The Jacobian from §15.10 is also a special case of this. There our surface S was in the flat plane, so the parametrization was of the form $\overrightarrow{r}(u, v) = \langle x(u, v), y(u, v), 0\rangle$. In that section we looked at integrals of scalar functions so the area element is $dS = |\overrightarrow{r}_u \times \overrightarrow{r}_v| \, du \, dv$. This is just the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

$$\det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{pmatrix} = \langle 0, 0, x_u y_v - x_v y_u\rangle$$

which has magnitude $|x_u y_v - x_v y_u| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

Stokes' theorem. §16.8

- Taking compatible orientations on closed curve C bounding surface S (meaning if we walk around C with our head in the direction of the normal, the surface is to the left):

$$\int\int_S \operatorname{curl} \overrightarrow{F} \cdot d\overrightarrow{S} = \int_C \overrightarrow{F} \cdot d\overrightarrow{r}$$

Divergence theorem. §16.9

- E is a solid enclosed by a closed surface S, with outward normal taken on S:

$$\int\int\int_E \operatorname{div}(\overrightarrow{F}) \, dV = \int\int_S \overrightarrow{F} \cdot d\overrightarrow{S}$$