QUALIFYING EXAM TRANSCRIPT FOR MORGAN WEILER, MAY 8 2015

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1-2:35. W: Can you give an example of a Hamiltonian, and say something about mechanics?

I fumbled for a bit, trying to say something about how the non-position variable was related to momentum. The goal was supposed to be that I say something about the equations of motion and/or conservation of energy. I did not really answer this question.

H: Blow up $\mathbb{CP}^2$ at $[0:0:1]$.

I did the complex blowup, gluing in a $\widetilde{\mathbb{C}}^2$ and the two projection maps Φ to $\mathbb{C}^2$ and pr to $\mathbb{CP}^1$ as in the first part of small McDuff-Salamon Chapter 7.

A: Why is $\mathbb{CP}^2$ symplectic and why is its blowup symplectic?

I wrote down $\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log(1+|z|^2)$, and got confused trying to explain how a $\frac{1}{z_i}$ in a coordinate change did not affect the form. I suggested that I could instead construct ω_{FS} using symplectic reduction but nobody wanted to see that.

For the symplectic-ness of the blowup, I wrote down $\omega(\lambda) = \Phi^{-1}\omega_0 + \lambda^2 pr\omega_{FS}$, where now ω_{FS} is the form on $\mathbb{CP}^1$ and claimed that $\Phi^* F^* \omega_0$ on $\mathbb{C}^2\backslash\{0\}$ was $\omega(\lambda)$, where $F : \mathbb{C}^2\backslash\{0\} \to \mathbb{C}^2\backslash B_0(\lambda)$ is given by $F(z) = \sqrt{|z|^2+\lambda^2}\frac{z}{|z|}$.

All: When is it possible to do this?

I said that $F \circ \Phi$ was a symplectomorphism between the δ-neighborhood of the zero set of $\widetilde{\mathbb{C}}^2 \xrightarrow{pr} \mathbb{CP}^1$ and $B_0(\sqrt{\delta^2+\lambda^2})\backslash B_0(\lambda)$, so we needed a Darboux chart in some ball in M on which the complex structure was also standard.

A: Can you say something about any additional compatibility?

I wrote down the definition of a Käher triple. I had to be prompted to say that J was supposed to be integrable.

H: What is $\mathbb{CP}^2$ blown up at a point topologically and how do you know this smooth manifold admits a symplectic form?

I wrote down $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, and tried to calculate its H^2 via Poincaré Duality and its H_2. I got confused about complex structures, symplectic structures, orientation, and coefficients, so this took a while. I said that a connect sum doesn't touch anything two-dimensional so the H_2 of the connect sum would be the direct sum of the H_2s of each part, and said something about cellular homology to calculate H_2 of each part. For some reason I thought for a while that the generators of each part would cancel each other out, but since they are completely different things there was no reason to think this.

W: What cohomology class is $\omega(\lambda)$ in?

I said it depended on λ but didn't remember precisely what to do.

H: What is the Delzant polytope of $\mathbb{CP}^2$?

At first I wrote down the wrong moment map but eventually fixed it with prompting.

W: What happens when you blow up?

I cut off a corner the size of which depended on λ. W reminded me that that was a way to see λ as the "weight" of the blowup.

2:40-3:40. A: Define the first Chern class.

I defined it as the obstruction to a section of the oriented circle bundle. I got confused about precisely how different choices of trivialization over a 1-skeleton resulted in a difference by a coboundary, did not actually show that the obstruction was closed, and realize now that I did not fully take orientations into account.

A: What is a $K(G, n)$? How can you use it?

I wrote the definition and had to be prompted to include restrictions on G, wrote the result about homotopy classes of maps from a space Y into a $K(G, n)$ being the same as $H^n(Y; G)$. I said something about how everything was unique and true when we restricted to CW complexes.

A: What happens if $n = 1$ and G is not abelian?

I wrote down the universal coefficients short exact sequence, said that Ext was zero since H_0 was zero if Y was connected, and fumbled around a bit before writing $[Y, K(G, 1)] \cong \mathrm{Hom}(\pi_1, G)$.

A: Give an example of a $K(Z, 2)$.

I used the long exact sequence of the fibration $S^1 \hookrightarrow S^\infty \to \mathbb{CP}^\infty$ to show that $\mathbb{CP}^\infty$ was a $K(Z, 2)$.

W: Give an example of a compact moduli space of J-holomorphic curves.

I wrote down the moduli space of curves in $\mathbb{CP}^2$ in the class $[\mathbb{CP}^1]$ with the standard complex structure and tried to argue that it was compact since $[\mathbb{CP}^1]$ attained the smallest possible $\omega(A)$ for $A \in H_2(\mathbb{CP}^2)$.

Much discussion ensued between W and H lamenting the state of affairs of "Gromov compactness" amongst the beginning symplectic students.

W: What is supposed to be compact?

I wrote down a statement of Gromov compactness, trying to follow her lecture notes (Math 278 Fall 2013).

W: What if u_∞ is constant?

I realized that I had to quotient my moduli space by $\mathrm{Aut}(\mathbb{CP}^1)$ to get compactness.

W: What are the key estimates for showing convergence on a compact set where the gradient is bounded? Assume it has an L^p bound for $p > 4$.

I wrote down the Δu bound and needed some help to write down the $||du||$ bound (hint: elliptic regularity gives a $||u||_{W^{k+2,p}}$ bound with $||\Delta u||_{W^{k,p}}$, so what kind of bound must we get on $||du||$?)

W: Why can't we use $p = \infty$?

I said something about L^1 and L^∞ not being reflexive so that the elliptic estimates would fail (the Sobolev embedding into continuous functions is still ok). I am still not quite sure what happens.